Limit cycles for fewnomial differential equations

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WORKSHOP ON DYNAMICAL SYSTEMS

Dedicated to Prof. Jaume Giné on occasion of his 60th birthday

Lleida, Thursday 11 – Friday 12 of January 2024

Talk based on the papers:

- A. Gasull, Chengzhi Li, J. Torregrosa. *Limit cycles for 3-monomial differential equations*. J. Math. Anal. Appl., **428**, 735–749. 2015.
- M. J. Álvarez, A. Gasull, R. Prohens. *Uniqueness of limit cycles for complex differential equations with two monomial*s. J. Math. Anal. Appl., 518, 126663. 2023.
- M.J. Álvarez, B. Coll, A. Gasull and R. Prohens. *More limit cycles for complex differential equations with three monomials*. In preparation 2024.

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Outline of the talk

1. Background and main results
2. Ideas of the proofs of the known results
3. Ideas of the proofs of the new results

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Background

Consider the differential equation:

$$\dot{z} = Az^k\bar{z}^l + Bz^m\bar{z}^n + Cz^p\bar{z}^q,$$

with k, l, m, n, p, q non-negative integers and $A, B, C \in \mathbb{C}$.

In the paper of Álvarez, Prohens and myelf it was proved that when $ABC = 0$ then its maximum number of limit cycles is 1.

In the 2015 paper, Chengzhi Li, J. Torregrosa, and myself it was proved that in general there is no upper bound for its number of limit cycles.

Background and aim of this talk

We rewrite the above results by using the following notation:

- $N = \max(k + l, m + n, p + q)$,
- $H_j(N) \in \mathbb{N} \cup \{\infty\}$ denotes the maximum number of limit cycles of the systems of the above type, with j monomials.

Theorem (AGP)

For $N = 1$, or N even, $H_2(N) = 0$ and for $N \geq 3$ odd, $H_2(N) = 1$.

Theorem (GLT)

For $N \geq 3$ odd, $$H_3(N) \geq \frac{N+3}{2}.$$

The aim of this talk is:

- Improve the lower bound of $H_3(N)$.
- Study $H_3(2)$.

The results are obtained in collaboration with Álvarez, Coll and Prohens.

New results about $H_3(N)$

Recall that it is known that for N odd, $H_3(N) \geq (N+3)/2$.

Theorem

For $N \geq 4$, $H_3(N) \geq N-3$.

Theorem

For $N = 4j-1$ and $j \geq 1$, $H_3(N) \geq N+1$.

All the above results give examples where each limit cycle surrounds a single critical point. For limit cycles with a different configuration we prove:

Proposition

For $N = 3j-1, j \geq 1$ there are equations with three monomials and $2j$ limit cycles (then $H_3(N) \geq \frac{2(N+1)}{3}$). The limit cycles are formed by j couples of two nested limit cycles surrounding, where each couple surrounds a single critical point.

Results about $H_3(2)$

There are $\binom{6}{3} = 20$ families of QS with 3 monomials. Among them it is well-known that the linear systems,

$$\dot{z} = A + Bz + C\bar{z},$$

and the homogenous QS,

$$\dot{z} = Az^2 + Bz\bar{z} + C\bar{z}^2$$

do not have limit cycles.

Hence it remains to study 18 families of QS, 9 of them with exactly one non-linear term and 9 with exactly two non-linear terms.

Our results about their number of limit cycles are resumed in next theorem.

Results about $H_3(2)$

Theorem

Consider the differential equation

$$\dot{z} = AM_1 + BM_2 + CM_3,$$

with $A, B, C \in \mathbb{C}$ *and* M_1, M_2 *and* M_3, *are 3 different fixed monomials* $M_j \in \{1, z, \bar{z}, z^2, z\bar{z}, \bar{z}^2\}$, *corresponding each one of the 18 families described above. Then its number of limit cycles is given in next tables.*

Results about $H_3(2)$

Monomials	$1, z$	$1, \bar{z}$	$z, \bar{z}$
z^2	0	≥ 1	≥ 1
$z\bar{z}$	1	1	1
$\bar{z}^2$	0	0	0

Monomials	$z^2, z\bar{z}$	$z^2, \bar{z}^2$	$z\bar{z}, \bar{z}^2$
1	$1+1$	$1+1$	$1+1$
z	≥ 1	≥ 2	≥ 1
$\bar{z}$	≥ 1	≥ 1	≥ 1

The $1+1$ means that the family has at most 2 limit cycles, that when they exist they are not nested.

Results about $H_3(2)$ and $H_3(3)$

Corollary

It holds that $H_3(2) \geq 2$ *and* $H_3(3) \geq 4$.

We have also proved that the maximum number of limit cycles of both families

$$\dot{z} = A + B\bar{z} + Cz^2 \quad \text{and} \quad \dot{z} = Az + B\bar{z} + Cz^2$$

coincide. As a consequence, the full case of 3-monomial QS with only one non-linear monomial showed in the first table would be totally solved if we were able to complete the study of the differential equation $\dot{z} = Az + B\bar{z} + Cz^2$.

Monomials	$1, z$	$1, \bar{z}$	$z, \bar{z}$
z^2	0	≥ 1	≥ 1
$z\bar{z}$	1	1	1
$\bar{z}^2$	0	0	0

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1. Background and main results
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Known results

To prove the old results people has used Abelian integrals, index theory, families of rotated vector fields, characterization of reversible centers, ...

The most difficult part of the proof for the 2 monomials case ($ABC = 0$) is to prove that many subcases do not have limit cycles and that the limit cycle only appears for

$$\dot{z} = Az^{l+1}\bar{z}^{l} + Bz^{n+1}\bar{z}^{n},$$

that has odd degree.

In the general case the key point has been the study of the quotient of two Abelian integrals and the presence of a rotational symmetry to produce limit cycles.

Some details of the proof of the 2015 result

Theorem (GLT)

For $N \geq 3$ odd, $H_3(N) \geq \dfrac{N+3}{2}$.

The result is a consequence of the following more concrete result:

Theorem

For $j \geq 3$, consider the 2-parameter family of systems

$$\dot{z} = (a+i)\,z - \frac{5i}{2}\,\bar{z}^{j-1} + (b+i)\ z^{j-1}\bar{z}^{j-2},$$

with $a, b \in \mathbb{R}$. Then there exist values for a and b for which the above equation has at least j limit cycles.

Notice that $N = 2j-3$ and the equation has $j = (N+3)/2$ limit cycles.

The differential equation

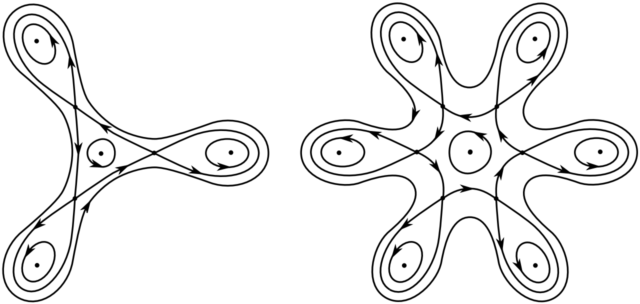
$$\dot{z} = (a+i)\,z - \frac{5i}{2}\,\bar{z}^{j-1} + (b+i)\,z^{j-1}\bar{z}^{j-2},$$

has rotational invariance of $2\pi/j$ radians. When $a=b=0$ the system is Hamiltonian, with Hamiltonian function

$$H(r,\theta) = \frac{r^2}{2} - \frac{5}{2j} r^j \cos(j\,\theta) + \frac{r^{2(j-1)}}{2(j-1)} - \tilde{\rho},$$

where $\tilde{\rho} = \frac{(j-2)(j-5)}{2j(j-1)}\,2^{\frac{2}{j-2}}$.

Their phase portraits are:



Centers when $a = b = 0$ for the cases $j = 3$ and $j = 6$.

Write $a = \varepsilon\,\alpha$ and $b = \varepsilon\,\beta$, for $\alpha, \beta \in \mathbb{R}$ with ε small enough. Then it can be seen the first order Melnikov function associated to the perturbed Hamiltonian has a simple zero and a limit cycle bifurcates from each period annulus.

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Proofs of the new results

The main tools for the proofs of the new results are:

- Computation of Lyapunov quantities, even when the linear part of the weak focus is not in normal form.
- Use of some non-invertible tranformations to increment the number of limit cycles.
- Changes of variable to transform some quadratic systems into other ones for which the maximum number of limit cycles is known.

Theorem

For $N \geq 4$, $H_3(N) \geq N - 3$.

PROOF: For each integer $n \geq 1$, let us consider the differential equation of degree $N = n + 3 \geq 4$,

$$\dot{z} = (A + B)z - Az^{n+1} - Bz^{n+2}\bar{z} = Az(1 - z^n) + Bz(1 - \bar{z}z^{n+1}),$$

being $A = n + 1 + a + i, B = -n + i$. The critical points of this equation are $z = 0$ and the points $z = w_j$ such that $w_j^n = 1$ for $j = 1, \ldots, n$.

Observe that this equation is invariant by the change of the dependent variable $u = w_j^{n-1}z$ for all $j = 1, \ldots, n$. By this change, the critical point w_j of the original equation is transformed into the critical point $u = 1$. Hence, varying j we get that all the critical points w_j of the original equation have the same character and stability as $z = 1$.
Let us study this critical point.

It holds that

$$\begin{aligned} \operatorname{div}(X)_{z=1} &= -2na, \\ \det(dX)_{z=1} &= n|A|^2 + n|B|^2 + n(n+1)|A||B| > 0. \end{aligned}$$

Hence, if $a = 0$ the point $z = 1$ is a weak focus.

Let us compute its first Lyapunov quantity L_1 and prove that $L_1 \neq 0$.

We perform the translation $w = z - 1$ to move the critical point to the origin and, for convenience, we change the sign of the vector field $(t \to -t)$. We arrive to the differential equation

$$\dot{w} = -(A+B)(w+1) + A(w+1)^{n+1} + B(w+1)^{n+2}(\bar{w}+1).$$

After some tedious computation we obtain that

$$L_1 = \frac{(5 + 2n - n^2)n^3}{9n^2 + 8n + 3}.$$

Notice that $L_1 > 0$ for $n = 1, 2, 3$ and $L_1 < 0$ for $n \geq 4$. Hence, because of the change of time, we know that the point $z = 1$ of the initial equation is an attractor when $n \leq 3$ and a repellor otherwise.

Finally it is easy to see that an Andronov-Hopf bifurcation undergoes, moving slightly the parameter a and taking it with the suitable sign.

One gets a hyperbolic limit cycle born from the critical point $(1,0)$ of the original differential equation.

From the symmetries of the initial differential equation, from each one of the n non-zero critical points of the system a limit cycle is born at the same time. Thus, the system has at least $n = N - 3$ hyperbolic limit cycles.

The limit cycles exist for $|a|$ small enough and $a < 0$ when $n = 1, 2, 3$ and are stable and also for $|a|$ small enough and $a > 0$ when $n \geq 4$ and are unstable.

Computation of the first Lyapunov quantity

We use, in real or complex coordinates respectively, the general expression of the first Lyapunov quantity L_1 (sometimes also called V_3) of the origin when it is a weak focus when it is not written in any special normal form.

The expressions are given in next slides.

We thank our colleague and friend Joan Torregrosa who gave us the key idea to compute it, and also all subsequent $L_j, j \geq 2$, by using a clever modification of Lyapunov procedure to find a local suitable Lyapunov function.

Theorem

Consider a $\mathcal{C}^4$ real planar differential equation

$$\begin{cases} \dot{x} = ux + vy + \sum_{j+k=2}^{3} a_{j,k} x^j y^k + O_4(x,y) = P(x,y), \\ \dot{y} = wx - uy + \sum_{j+k=2}^{3} b_{j,k} x^j y^k + O_4(x,y) = Q(x,y), \end{cases}$$

where $w > 0$, $u^2 + vw < 0$, and $O_4(x,y)$ denotes terms of order at least 4. Then the origin is a weak focus and its first Lyapunov quantity is

$$L_1 = \frac{L}{4u^2 + 3v^2 - 2vw + 3w^2}, \quad \textit{where}$$

$$\begin{aligned} L = \; & \big(a_{1,1}a_{2,0} - b_{0,2}b_{1,1}\big)(2u^2 - vw) \\ & + \left(a_{1,1}b_{2,0} - 2{a_{2,0}}^2 + a_{2,0}b_{1,1} + 2b_{0,2}b_{2,0} + {b_{1,1}}^2\right) uv \\ & + \left(2a_{0,2}a_{2,0} + a_{0,2}b_{1,1} + {a_{1,1}}^2 + a_{1,1}b_{0,2} - 2{b_{0,2}}^2\right) uw \\ & - b_{2,0}\left(2a_{2,0} + b_{1,1}\right) v^2 + a_{0,2}\left(a_{1,1} + 2b_{0,2}\right) w^2 \\ & - \big(2\left(a_{2,1} + b_{1,2}\right) u - \left(3a_{3,0} + b_{2,1}\right) v + \left(a_{1,2} + 3b_{0,3}\right) w\big)(u^2 + vw). \end{aligned}$$

Theorem

Consider the $\mathcal{C}^4$ *differential equation*

$$\dot{z} = Rz + S\bar{z} + Az^2 + Bz\bar{z} + C\bar{z}^2 + Dz^3 + Ez^2\bar{z} + Fz\bar{z}^2 + G\bar{z}^3 + O_4(z,\bar{z})$$

where all the involved parameters are complex, $R = r_1 + \mathrm{i}r_2$, $S = s_1 + \mathrm{i}s_2$. *When* $r_1 = 0$, $S\bar{S} - R\bar{R} < 0$ *and* $\mathrm{Im}(R+S) > 0$ *the origin is a weak focus and its first Lyapunov quantity is*

$$L_1 = \frac{\mathrm{Im}(M)}{2R\bar{R} + S\bar{S}},$$

where

$$\begin{aligned} M = {} & \big(2RE - S(3D+\bar{F})\big)(R\bar{R} - S\bar{S}) + \big(AB + 2\bar{A}C + BC\big)\bar{S}^2 \\ & + \big(2AC - 2A^2 + \bar{A}B + B^2 + \bar{B}C\big)R\bar{S} - A^2(S-\bar{S})(R-\bar{R}) \\ & - AB\big(2R\bar{R} + S\bar{S} + \bar{S}^2\big). \end{aligned}$$

Recovering the classical expression of L_1

$$\dot{z} = Rz + S\bar{z} + Az^2 + Bz\bar{z} + C\bar{z}^2 + Dz^3 + Ez^2\bar{z} + Fz\bar{z}^2 + G\bar{z}^3 + O_4(z,\bar{z}).$$

$$L_1 = \frac{\operatorname{Im}(M)}{2R\bar{R} + S\bar{S}}, \quad \text{where}$$

$$M = \big(2RE - S(3D + \bar{F})\big)(R\bar{R} - S\bar{S}) + \big(AB + 2\bar{A}C + BC\big)\bar{S}^2 + \big(2AC - 2A^2 + \bar{A}B + B^2 + \bar{B}C\big)R\bar{S} - A^2(S - \bar{S})(R - \bar{R}) - AB\big(2R\bar{R} + S\bar{S} + \bar{S}^2\big).$$

- Notice that when the origin is a weak focus written in normal form, that is $R = \mathrm{i}$ and $S = 0$, then

$$L_1 = \operatorname{Re}(E) - \operatorname{Im}(AB),$$

  a well-known and nice expression of L_1.
- The bifurcation of Andronov–Hopf happens when instead of $r_1 = \operatorname{Re}(R) = 0$ we take $|r_1| \neq 0$ small enough and $\operatorname{Re}(R)\operatorname{Im}(M) < 0$.

Computation of the first Lyapunov quantity.

IDEA OF THE COMPUTATIONS: Consider $H(x,y) = \sum_{k\geq 2} H_k(x,y)$, where

$$H_2(x,y) = -\frac{v}{2}y^2 - uxy + \frac{w}{2}x^2,$$

and H_k are homogeneous polynomials of degree k. Notice that H_2 is a first integral of the linear part of the system which corresponds to a center and it is positive definite.

Then this small variation of the Lyapunov's method consists in proving that there exist $H_k, k \geq 3$ (not unique), such that

$$H_x(x,y)P(x,y) + Q(x,y)H_y(x,y) = \sum_{m=1}^{M} L_m(x^2+y^2)^{m+1} + O_{2M+3}(x,y),$$

for a suitable M, where recall that $H_x P + H_y Q = \dot{H}$.

The expressions L_m are the Lyapunov quantities.

Proof of the other results about $H_3(N)$

Theorem

For $N \geq 4j - 1$ and $j \geq 1$, $H_3(N) \geq N + 1$.

- The proof of the theorem starts with a cubic system $(N = 3)$ with 4 limit cycles each one of them surrounding a single critical point.

Proposition

For $N = 3j - 1, j \geq 1$ there are equations with three monomials and $2j$ limit cycles (then $H_3(N) \geq \frac{2(N+1)}{3}$). The limit cycles are formed by j couples of two nested limit cycles surrounding, where each couple surrounds a single critical point.

- The proof of the proposition starts by bifurcating 2 limit cycles from a weak focus of a quadratic system $(N = 2)$ by a codimension two Andronov-Hopf bifurcation.

Study of some quadratic cases

- All examples with limit cycles (the lower bounds) of both tables are realized via Andronov-Hopf type bifurcations. We skip the details.
- Let us prove for instance the red cases of next tables:

Monomials	$1, z$	$1, \bar{z}$	$z, \bar{z}$
z^2	0	≥ 1	≥ 1
$z\bar{z}$	1	1	1
$\bar{z}^2$	0	0	0

Monomials	$z^2, z\bar{z}$	$z^2, \bar{z}^2$	$z\bar{z}, \bar{z}^2$
1	$1+1$	$1+1$	$1+1$
z	≥ 1	≥ 2	≥ 1
$\bar{z}$	≥ 1	≥ 1	≥ 1

Study of some quadratic cases

Monomials	$1, z$	$1, \bar{z}$	$z, \bar{z}$
$\bar{z}^2$	0	0	0

This result is a straightforward consequence of Bendixson–Dulac criterion because if $\dot{z} = F(z, \bar{z})$ the divergence of the associated vector field is

$$2\,\mathrm{Re}\left(\frac{\partial}{\partial z} F(z, \bar{z})\right),$$

and for the differential equations

$$\dot{z} = A + Bz + C\bar{z}^2, \quad \dot{z} = A + B\bar{z} + C\bar{z}^2, \quad \dot{z} = Az + B\bar{z} + C\bar{z}^2,$$

the respective divergences are $2\,\mathrm{Re}(B), 0$ and $2\,\mathrm{Re}(A)$. Because they do not change sign, the differential equations do not have limit cycles.

Study of some quadratic cases

Monomials	$z^2, z\bar{z}$	$z^2, \bar{z}^2$	$z\bar{z}, \bar{z}^2$
1	$1+1$	$1+1$	$1+1$

The proof is based on next result, proved in 1981 by Suo Guangjian and published in Chinese. We include here a proof inspired by the one of the original paper.

Theorem (Suo Guangjian)

The system

$$\dot{z} = A + Bz^2 + Cz\bar{z} + D\bar{z}^2$$

either does not have limit cycles or it has exactly two limit cycles, γ and $-\gamma$. Moreover, in this latter case they are hyperbolic, with different stabilities and each one of them surrounds a different critical point.

A preliminary result

The following theorem is a well-known result on QS. We state next version due to Coppel:

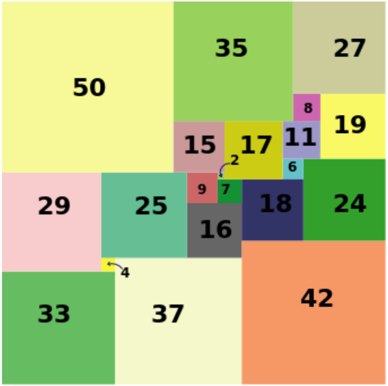
Theorem

Suppose a QS satisfies one of the following conditions:

- *it has an invariant straight line,*
- *the highest degree terms are proportional,*

Then, the QS has at most one limit cycle and when it exists it is hyperbolic.

Proofs without words



$$112^2 = 2^2 + 4^2 + 6^2 + 7^2 + 8^2 + 9^2 + 11^2 + 15^2 + 16^2 + 17^2 + 18^2 + 19^2 + 24^2 + 25^2 + 27^2 + 29^2 + 33^2 + 35^2 + 37^2 + 42^2 + 50^2$$

A proof without words of Guangjian's result

$$\dot z = A + Bz^2 + Cz\bar z + D\bar z^2.$$

$$\Downarrow$$

$$\begin{cases}\dot x = a + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2,\\ \dot y = b + b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2.\end{cases}$$

$$\Downarrow \quad (x_0, y_0) \longrightarrow (1,0)$$

$$\begin{cases}\dot x = a - ax^2 + a_{1,1}xy + a_{0,2}y^2,\\ \dot y = b - bx^2 + b_{1,1}xy + b_{0,2}y^2.\end{cases}$$

$$\Downarrow \quad \text{A “rotation”}$$

$$\begin{cases}\dot x = a - ax^2 + a_{1,1}xy,\\ \dot y = b - bx^2 + b_{1,1}xy + b_{0,2}y^2.\end{cases}$$

$$\Downarrow$$

A proof without words of Guangjian's result

$$\begin{cases} \dot{x} = a - ax^2 + a_{1,1}xy, \\ \dot{y} = b - bx^2 + b_{1,1}xy + b_{0,2}y^2. \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{x} = 1 - x^2 + a_{1,1}xy, \\ \dot{y} = b - bx^2 + b_{1,1}xy + b_{0,2}y^2. \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{x} = 1 - x^2 + xy, \\ \dot{y} = b - bx^2 + b_{1,1}xy + b_{0,2}y^2. \end{cases}$$

$$\Downarrow \quad (X = x^2, \quad Y = 1 - x^2 + xy)$$

$$\begin{cases} X' = & 2XY, \\ Y' = & b_{0,2} + (b - 2\,b_{0,2} - b_{1,1})\,X - (2\,b_{0,2} + 1)\,Y + (-b + b_{0,2} + b_{1,1})\,X^2 \\ & + (2\,b_{0,2} + b_{1,1} - 1)\,XY + (b_{0,2} + 1)\,Y^2. \end{cases}$$

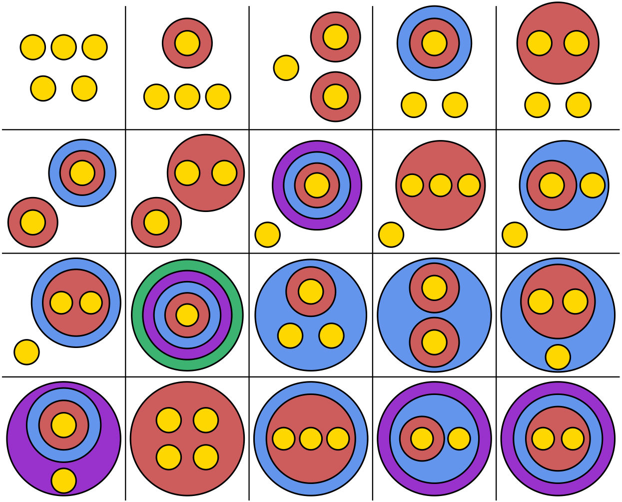


Figure from: *Counting configurations of limit cycles and centers*, A. Gasull, A. Guillamon, V. Mañosa, 2023.

Thank you very much for your attention....

... but this is not the end

A strong relation with Lleida group

All started with the visits of Jaume Llibre and myself to Javier Chavarriga.

Very soon Jaume Giné joined us.

Along the years we have had a very fruitful collaboration.

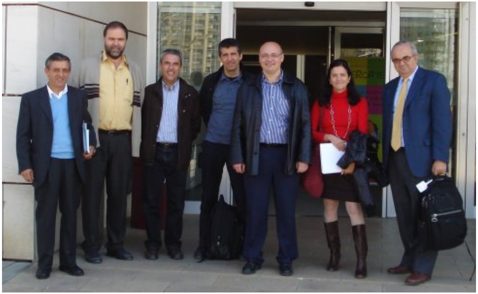
Joint papers: 2007, 2010, 2016, 2017, 2017, 2020

- A. Gasull, J. Giné, C. Valls. *Highest weak focus order for trigonometric Liénard equations*. Ann. Mat. Pur. Appl., **199**, 1673–1684. 2020.
- A. Gasull, J. Giné. *Integrability of Liénard systems with a weak saddle*. Z. Angew. Math. Phys., **68:** (13), 2017.
- A. Gasull, J. Giné, C. Valls. *Center problem for trigonometric Liénard systems*. J. Differential Equations, **263**, 3928–3942. 2017.
- A. Gasull, J. Giné, J. Torregrosa. *Center problem for systems with two monomial nonlinearities*. Commun. Pure Appl. Anal., **15**(2), 577–598. 2016.
- A. Gasull, J. Giné. *Cyclicity versus center problem*. Qual. Theory Dyn. Syst., **9**(1-2), 101–111. 2010.
- A. Gasull, J. Giné, M. Grau. Multiplicity of limit cycles and analytic m-solutions for planar differential systems. J. Differential Equations, **240**, 375-398. 2007.

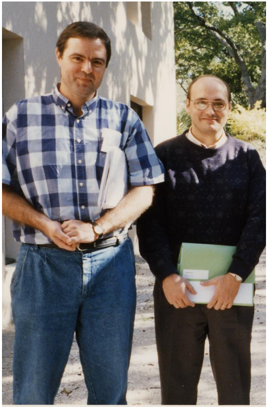
Many joint experiences

Many joint experiences

Many joint experiences



Feliç 60 aniversari, Jaume!



Luminy 1997

Cádiz 2018

Encara que ja no som gaire joves, espero poder mantenir la nostra col·laboració molts més anys!