Workshop on Dynamical Systems

3-dimensional smooth and piecewise smooth vector fields with invariant spheres

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1 Buzzi, C. A., Rodero, A. L. and Torregrosa, J. *Center and limit cycles for piecewise linear and quadratic vector fields on invariant spheres*, Journal of Nonlinear Science, 31, 92 (2021).

2 Buzzi, C. A., Rodero, A. L. and Torregrosa, J. 3-*dimensional piecewse linear and quadratic vector fields with invariant speres*. Preprint (2023)

Suppose $X : \mathbb{R}^3 \to \mathbb{R}^3$ that admits $H(x, y, z) = x^2 + y^2 + z^2$ as first integral. It means that all the spheres

$$\mathbb{S}^2_\rho = \{(x, y, z) : x^2 + y^2 + z^2 = \rho^2\}$$

are invariant by the flow of X. We denote by $\mathfrak{X}$ this class of smooth vector fields and by $\mathfrak{X}_n$ when they are polynomials of degree n.

Consider the piecewise 3-dimensional differential vector fields with the separation set $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$, that is

$$Y(x, y, z) = \begin{cases} X^+(x, y, z), & z \geq 0, \\ X^-(x, y, z), & z \leq 0, \end{cases} \tag{1}$$

such that $X^+, X^- \in \mathfrak{X}$. We denote this class by $\mathcal{X}$ and by $\mathcal{X}_n$ when $X^+, X^- \in \mathfrak{X}_n$.

We follow the Filippov convention in Σ.

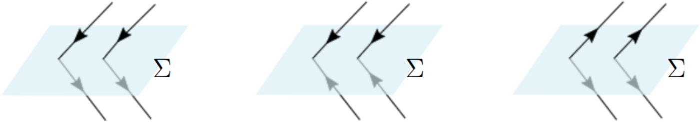


Figure: Crossing, sliding and escaping regions, respectively.

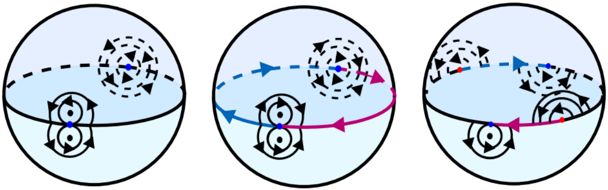


Figure: Examples of piecewise smooth vector fields on invariant spheres.

3 Filippov, A. F. *Differential equations with discontinuous righthand sides*, vol. 18 of Mathematics and its Applications (Soviet Series), Springer, Dordrecht, 1988. Originally published in Russian.

Some properties of smooth vector fields $X \in \mathfrak{X}$

Lemma.

Let M be an orthogonal matrix (i.e. $M^{-1} = M^t$). If $X \in \mathfrak{X}$ then $M \cdot X(M^t) \in \mathfrak{X}$.

Remark

One of the equilibrium points of $X \in \mathfrak{X}$, on the sphere $\mathbb{S}^2_\rho$, can be always located at $(0, \rho, 0)$ using an orthogonal change of coordinates.

Lemma.

The homogeneity property is invariant by an orthogonal change of coordinates.

Lemma.

Let $X \in \mathfrak{X}^H$. The phase portrait on each sphere is topologically equivalent to the one on the sphere of radius 1. Moreover, X has a straight line passing through the origin filled of equilibrium points.

Proof.

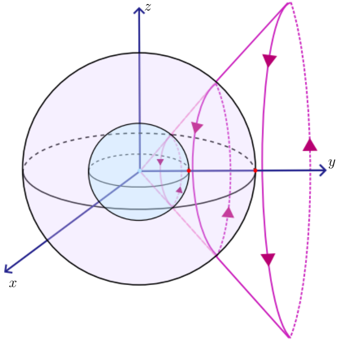
The proof follows just doing the change of coordinates $\mathrm{y} = \mathrm{x}/\rho$ and a time rescaling if necessary. □

The geometric main idea

We consider the vector fields $X \in \mathfrak{X}$ restrict to an invariant sphere. When X is homogeneous, the phase portrait on each sphere is topologically equivalent to the one on the sphere of radius 1. So,

if $X_{|_{\mathbb{S}^2_1}}$ has a limit cycle on $\mathbb{S}^2_1$, then X has an invariant cone.

The same idea is valid for piecewise smooth vector fields $Y \in \mathcal{X}$.



The geometric main idea

On the other hand, suppose that:

- The phase portrait of $X_{|_{\mathbb{S}^2_\rho}}$ is not always topologically equivalent to the phase portrait of $X_{|_{\mathbb{S}^2_1}}$;
- X has a hyperbolic limit cycle on $\mathbb{S}^2_1$.

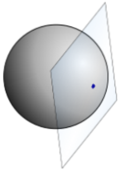
Since this limit cycle is hyperbolic on $\mathbb{S}^2_1$, it is normally hyperbolic with respect to the radial direction. Then X has a invariant surface foliated by these limit cycles. The global structure of each invariant surface is due to the birth or death of limit cycles.

Remark

The same idea is valid for piecewise smooth vector fields $Y \in \mathcal{X}$.

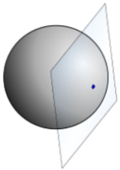
Let $\mathfrak{p} : \mathbb{S}^2_\rho \setminus \{(0, -\rho, 0)\} \to \mathbb{R}^2$ be the stereographic projection on the plane $\{(x, y, z) \in \mathbb{R}^3 : y = \rho\}$ given by

$$\mathfrak{p}(x, y, z) = \left(\frac{2\rho x}{y + \rho}, \frac{2\rho z}{y + \rho} \right).$$



Let $\mathfrak{p} : \mathbb{S}^2_\rho \setminus \{(0,-\rho,0)\} \to \mathbb{R}^2$ be the stereographic projection on the plane $\{(x,y,z) \in \mathbb{R}^3 : y = \rho\}$ given by

$$\mathfrak{p}(x,y,z) = \left(\frac{2\rho x}{y+\rho}, \frac{2\rho z}{y+\rho} \right).$$



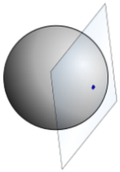
Then the projection $\mathcal{P}_X : \mathbb{R}^2 \to \mathbb{R}^2$ of the vector field X writes as

$$\mathcal{P}_X(\mathrm{u}) = d\mathfrak{p}_{\mathfrak{p}^{-1}(\mathrm{u})} \circ X \circ \mathfrak{p}^{-1}(\mathrm{u}), \tag{2}$$

where $X = X_{|_{\mathbb{S}^2_\rho}}$, $\mathrm{u} = (u,v)$. Note that $\mathfrak{p}(0,\rho,0) = (0,0)$.

Let $\mathfrak{p} : \mathbb{S}^2_\rho \setminus \{(0,-\rho,0)\} \to \mathbb{R}^2$ be the stereographic projection on the plane $\{(x,y,z) \in \mathbb{R}^3 : y = \rho\}$ given by

$$\mathfrak{p}(x,y,z) = \left(\frac{2\rho x}{y+\rho}, \frac{2\rho z}{y+\rho}\right).$$



$\mathfrak{p}$ sends the separation set $\Sigma = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$ of a piecewise vector field $Y \in \mathcal{X}$ to $\{(u,v) \in \mathbb{R}^2 : v = 0\}$. Thus, the projection $\mathcal{P}_Y : \mathbb{R}^2 \to \mathbb{R}^2$ of (1) is written as

$$\mathcal{P}_Y(\mathrm{u}) = \begin{cases} \mathcal{P}_{X^+}(\mathrm{u}), & v \geq 0, \\ \mathcal{P}_{X^-}(\mathrm{u}), & v \leq 0, \end{cases} \tag{2}$$

where $X^\pm = X^\pm_{|_{\mathbb{S}^2_\rho}}$, $\mathrm{u} = (u,v)$.

We use the Lyapunov constants and the integrability to study the center and cyclicity problems. These are widely used and well-known tools. So, for the sake of time, we will briefly recall the piecewise method.

4 Andronov, A. A., Leontovich, E. A., Gordon, I. I. and Maĭer, A. G. *Theory of bifurcations of dynamic systems on a plane*. Halsted Press [A division of John Wiley Sons], New York-Toronto, Ont.; Israel Program for Scientific Translations, Jerusalem-London, 1973. Translated from the Russian.

5 Christopher, C. *Estimating limit cycle bifurcations from centers*. In Differential equations with symbolic computation. Springer, 2005, pp. 23–35.

6 Dumortier, F., Llibre, J., and Artés, J. C *Qualitative theory of planar differential systems*. Universitext. Springer-Verlag, Berlin, 2006.

Lyapunov constants and local cyclicity

We will recall the stability algorithm for planar piecewise smooth vector fields of the form

$$Y(x,y) = \begin{cases} X^{+}(x,y),\ y \geq 0, \\ X^{-}(x,y),\ y \leq 0, \end{cases} \tag{3}$$

having both $X^{\pm}$ an equilibrium point of nondegenerate center-focus type at the origin. That is,

$$X^{\pm}(x,y) = \left(\alpha^{\pm}x - \beta^{\pm}y + \sum_{k=2}^{n} P_k^{\pm}(x,y), \beta^{\pm}x + \alpha^{\pm}y + \sum_{k=2}^{n} Q_k^{\pm}(x,y) \right),$$

with $P_k^{\pm}$ and $Q_k^{\pm}$ homogeneous polynomials of degree k in the variables x and y. We have assumed that both linear parts are in Jordan's normal form.

We assume $\beta^{\pm} \neq 0$ as the non degeneracy condition for each $X^{\pm}$. Using polar coordinates, $(x, y) = (r\cos\theta, r\sin\theta)$, we write system (3) as

$$\begin{cases} \dot{r} = R^{+}(r,\theta), & \theta \in [0,\pi], \\ \dot{r} = R^{-}(r,\theta), & \theta \in [\pi, 2\pi], \end{cases}$$

where the dot represents the derivative with respect to θ.
Consider $r^{\pm}(\theta, r_0)$ the solution of $\dot{r} = R^{\pm}(r,\theta)$ with initial condition $r^{\pm}(0, r_0) = r_0$.
For $r_0 > 0$ sufficiently small, the expansion in Taylor's series of $r^{\pm}(\theta, r_0)$ is given by

$$r^{\pm}(\theta, r_0) = r_0 + \sum_{k=1}^{\infty} r_k^{\pm}(\theta) r_0^k,$$

with $r_k^{\pm}(0) = 0$, for all $k \geq 1$, r^{+} defined for $\theta \in [0,\pi]$ and r^{-} defined for $\theta \in [\pi, 2\pi]$.

The Poincaré half-return maps are defined by

$$\Pi^+(r_0) = r^+(\pi, r_0) \text{ and } \widetilde{\Pi}^-(r_0) = r^-(-\pi, r_0),$$

where $\widetilde{\Pi}^-$ denotes the inverse of Π^- since both $r^\pm$ are defined with initial condition $\theta = 0$ and $r_0 > 0$ is sufficiently small.
The displacement function, which is an analytic function, is given by

$$\Delta(r_0) = \widetilde{\Pi}^-(r_0) - \Pi^+(r_0) = \sum_{k=1}^{\infty} L_k r_0^k.$$

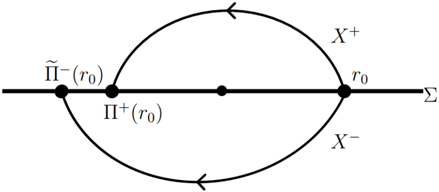


Figure: Displacement function

When $\alpha^+\alpha^- \neq 0$ the origin is a hyperbolic equilibrium point, so we assume $\alpha^+\alpha^- = 0$ on the following.

When, $L_1 = 0$ and, for $k \geq 2$, we can define the k-th Lapunov constant by $L_k \neq 0$, when $L_1 = \cdots = L_{k-1} = 0$. In this case, if there exists $k \geq 2$ so that $L_k \neq 0$, then the origin of system (3) is a weak focus of order k. Otherwise the origin is a center.

Moreover, for piecewise smooth vector fields we can obtain one limit cycle moving the equilibrium points on Σ. Because a sliding or escaping segment is created adding adequately some perturbative parameters. This is known as a pseudo-Hopf bifurcation.

So, we can obtain k crossing limit cycles from a weak-focus of order k.

[7] Gasull, A., Torregrosa, J. *Center-focus problem for discontinuous planar differential equations*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 13 (2003), 1755-1765 (Dynamical systems and functional equations - Murcia, 2000).

[8] Castillo, J., Llibre, J., and Verduzco, F., *The pseudo-Hopf bifurcation for planar discontinuous piecewise linear differential systems*, Nonlinear Dynamics, 90 (2017), pp. 1829–1840.

Linear vector fields

Let $X : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear vector field

$$\begin{aligned}\dot{x} &= b_0 + b_1 x + b_2 y + b_3 z,\\ \dot{y} &= c_0 + c_1 x + c_2 y + c_3 z,\\ \dot{z} &= d_0 + d_1 x + d_2 y + d_3 z.\end{aligned}$$

Suppose that $< \mathrm{x}, X(\mathrm{x}) >= 0$, for all x (it means that $X \in \mathfrak{X}_1$). Then it is always homogeneous and it writes in the form

$$\begin{aligned}\dot{x} &= -a_1 y - a_2 z,\\ \dot{y} &= a_1 x - a_3 z,\\ \dot{z} &= a_2 x + a_3 y.\end{aligned} \tag{4}$$

Theorem

Let $p \in \mathbb{S}^2_\rho = \{(x, y, z) : x^2 + y^2 + z^2 = \rho^2\}$ be an equilibrium point of system (4) which is isolated in $\mathbb{S}^2_\rho$. Then p is a center. Moreover, the system is completely integrable.

Proof.

$$H_2(x, y, z) = a_3 x - a_2 y + a_1 z$$

is also a first integral for the linear sistem (4). □

Remark

It means that no differential system $X \in \mathfrak{X}_1$ admits an isolated invariant cone.

Any $Y = (X^+, X^-) \in \mathcal{X}_1$ is a piecewise 3-dimensional vector field of the form

$$Y(x,y,z) = \begin{cases} X^+(\mathrm{x}; a_1^+, a_2^+, a_3^+), & z \geq 0, \\ X^-(\mathrm{x}; a_1^-, a_2^-, a_3^-), & z \leq 0, \end{cases} \tag{5}$$

with

$$X^{\pm}(\mathrm{x}; a_1^{\pm}, a_2^{\pm}, a_3^{\pm}) = (-a_1^{\pm} y - a_2^{\pm} z, a_1^{\pm} x - a_3^{\pm} z, a_2^{\pm} x + a_3^{\pm} y). \tag{6}$$

Theorem

No piecewise differential system $Y \in \mathcal{X}_1$, given by (5), admits an isolated invariant cone.

It is equivalent to prove that:

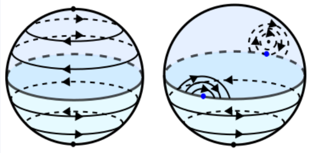
Theorem

No piecewise differential system $Y \in \mathcal{X}_1$, given by (5), admits limit cycles on the sphere $\mathbb{S}^2_\rho$.

Proof.

The key point of this proof is understanding how the level curves of $H_2^{\pm}(x,y,z) = a_3^{\pm}x - a_2^{\pm}y + a_1^{\pm}z$ interact with Σ.

Firstly we note that when $a_2^{\pm} = a_3^{\pm} = 0$, $H_2^{\pm}(x,y,z) = a_1 z$, which implies that Σ is invariant by the flow of (6). Then, on the following we assume $(a_2^{\pm})^2 + (a_3^{\pm})^2 \neq 0$



Let $p = (x_0, y_0, 0) \in \Sigma \cap \mathbb{S}^2_\rho$. Then, there exist $k^{\pm}$ such that $H_2^{\pm}(p) = k^{\pm}$. The half-return maps $\pi^{\pm}(p) = q^{\pm} = (x_1^{\pm}, y_1^{\pm}, 0)$ satisfy

$$
\begin{aligned}
H(q^{\pm}) &= \rho^2, \\
H_2^{+}(q^{+}) &= a_3^{+}x_1^{+} - a_2^{+}y_1^{+} = k^{+}, \\
H_2^{-}(q^{-}) &= a_3^{-}x_1^{-} - a_2^{-}y_1^{-} = k^{-}.
\end{aligned}
$$

Proof.

Solving the systems of equations

$$\{H(q^+) = \rho^2, H_2^+(q^+) = k^+\}, \{H(q^-) = \rho^2,\ H_2^-(q^-) = k^-\}$$

we obtain the solutions

$$q^{\pm} = \left(-\frac{((a_2^{\pm})^2 - (a_3^{\pm})^2)x_0 + 2a_2^{\pm}a_3^{\pm}y_0}{(a_2^{\pm})^2 + (a_3^{\pm})^2}, \frac{((a_2^{\pm})^2 - (a_3^{\pm})^2)y_0 - 2a_2^{\pm}a_3^{\pm}x_0}{(a_2^{\pm})^2 + (a_3^{\pm})^2}, 0 \right).$$

So, the difference map, $d(p) = \pi^+(p) - \pi^-(p) : \Sigma \to \mathbb{R}$, is such that

$$\begin{aligned} d(p) = &(2(a_2^- a_3^+ - a_3^- a_2^+)((a_2^- a_3^+ + a_2^+ a_3^-)x_0 - (a_2^- a_2^+ - a_3^- a_3^+)y_0), \\ &- 2(a_2^- a_3^+ - a_3^- a_2^+)((a_2^- a_2^+ - a_3^- a_3^+)x_0 + (a_2^- a_3^+ + a_2^+ a_3^-)y_0), 0). \end{aligned}$$

Consequently, the difference map $d(p)$ is identical to zero if, and only if, $a_2^+ a_3^- = a_2^- a_3^+$. Hence, either all the crossing trajectories of (5), on $\mathbb{S}^2_\rho$, are closed or none of them are. □

Quadratic homogeneous vector fields

Let $X \in \mathfrak{X}_2^H$, without lose of generality, it writes in the following canonical form

$$\begin{aligned} \dot{x} &= -a_4xy - a_5xz - (a_6 + a_7)yz - a_8z^2, \\ \dot{y} &= a_4x^2 + a_6xz - a_9z^2, \\ \dot{z} &= a_5x^2 + a_7xy + a_8xz + a_9yz. \end{aligned} \tag{7}$$

We notice that the equilibrium point is located at $(0, 1, 0)$.

Theorem

The equilibrium point $(0,1,0)$ *of system* (7) *is a nondegenerate center if, and only if,* $a_7 \neq 0$, $a_4 = a_9$, *and* $a_4 a_5 a_8 a_9 + a_5 a_6 a_7 a_8 + a_5^2 a_7 a_9 + a_5 a_8 a_9^2 - a_7 a_8^2 a_9 = 0$.

Proof.

The projected system $\mathcal{P}_X$ is of the form

$$\begin{aligned} \dot{u} &= -4a_4 u - 4\xi v - 4a_5 uv - 4a_8 v^2 - a_4 u^3 - (\xi - 2a_7)u^2 v + (a_4 + 2a_9)uv^2 + \xi v^3, \\ \dot{v} &= 4a_7 u + 4a_9 v + 4a_5 u^2 + 4a_8 uv - a_7 u^3 - (2a_4 + a_9)u^2 v - (2\xi - a_7)uv^2 + a_9 v^3, \end{aligned} \tag{8}$$

where $\xi = (w^2 + a_4 a_9)/a_7$ and $w^2 = a_6 a_7 + {a_7}^2 - {a_9}^2$. It is easy to check that the trace and determinant of J are $-4(a_4 - a_9)$ and $16w^2$, respectively. Moreover,

$$L_1 = \frac{16(a_7^2 + a_9^2 + w^2)C}{3(a_9^2 + w^2)^2},$$

where $C = -a_5^2 a_7 a_9 + a_5 a_7^2 a_8 - a_5 a_8 a_9^2 - a_5 a_8 w^2 + a_7 a_8^2 a_9$.
We finish the proof showing that under these conditions system (7) is always time reversible. □

Proposition

The quadratic homogeneous vector field (7) *has at least one limit cycle bifurcating from* $(0, 1, 0)$ *on the sphere* $\mathbb{S}_1^2$.

Proof.

Consider the quadratic homogeneous vector field (7) and its projection (8) with the parameters values $(a_4, a_5, a_7, a_8, a_9, w) = (1 + \varepsilon, 1, 1, 0, 1, 1)$, given by

$$\begin{aligned} \dot{u} =& (-4 + \varepsilon)u - 4(2 + \varepsilon)v - 4uv - (1 + \varepsilon)u^3 + \varepsilon u^2 v + (3 + \varepsilon)uv^2 + (2 + \varepsilon)v^3, \\ \dot{v} =& 4u + 4v + 4u^2 - u^3 - (3 + \varepsilon)u^2 v - (3 + \varepsilon)uv^2 + v^3. \end{aligned} \tag{9}$$

Note that the origin is an equilibrium point of (9). Let J be the Jacobian matrix associated to (9) at the origin. As the trace of J is ε and its determinant is $16+12\varepsilon$, then the origin is a weak focus for $\varepsilon = 0$. The prof follows by the classical Hopf bifurcation.

□

On the following we will focus our attention on the center-focus problem that appears naturally for the piecewise smooth system

$$Y(x,y,z) = \begin{cases} X^+(x,y,z), \ z \geq 0, \\ X^-(x,y,z), \ z \leq 0, \end{cases} \tag{10}$$

where we obtain $X^{\pm}$ doing $a_i = a_i^{\pm}$ in (7) and assuming that

$$p = (0,1,0) \in \Sigma = \{(x,y,z) \in \mathbb{R}^3 : \ z = 0\}$$

is of the center type for both X^+ and X^- on $\mathbb{S}^2_\rho$.

We also assume that the system (10) and the projected associated systems $\mathcal{P}_Y = (\mathcal{P}_{X^+}, \mathcal{P}_{X^-})$ is continuous but not differentiable on the separation set Σ. It occurs if, and only if, $a_4^- = a_4^+$, $a_5^- = a_5^+$, and $a_7^- = a_7^+$. Under these assumptions we calculated the Lyapunov constants and we obtain the following result.

Proposition

The piecewise continuous vector field (10) *has a center at the equilibrium point* $(0,1,0)$, *on* $\mathbb{S}_1^2$, *if* $a_7^{\pm} \neq 0$, $a_4^{\pm} = a_9^{\pm}$ *and one of the following conditions is satisfied:*

- a $a_8^- = -a_8^+$, $a_9^- = 0$, *and* $w^+ = w^-$;
- b $a_7^- = \pm w$, $a_9^- = 0$, *and* $w^+ = w^-$;
- c $a_8^+ = a_8^-$,
  $-(a_5^-)^2 a_7^- a_9^- + a_5^-(a_7^-)^2 a_8^- - a_5^- a_8^- (a_9^-)^2 - a_5^- a_8^- w^2 + a_7^- (a_8^-)^2 a_9^- = 0$, *and*
  $w^+ = w^-$;
- d $a_5^- = 0$ *and* $a_9^- = 0$.

Proposition

Consider system (10) with $a_5^- = 1, a_7^- = 1, a_8^+ = 3, a_8^- = 1, a_9^- = 0$, and $w^+ = w^- = 2$. Then, the equilibrium point $p = (0,1,0)$ is a weak focus of third-order and there exist 2 small amplitude limit cycles, on $\mathbb{S}_1^2$, bifurcating from p with a continuous perturbation in $\mathcal{X}_2^H$.

Proof.

For these values of parameters, we have

$$L_2 = 0 \text{ and } L_3 = 15\pi/16 \neq 0.$$

Hence, adding the trace parameter and using the derivation-division algorithm we obtain 2 small amplitude crossing limit cycles bifurcating from the equilibrium point $(0,1,0)$ on $\mathbb{S}_1^2$. □

With the previous results we can see an important difference between linear and quadratic homogeneous vector fields in the classes $\mathfrak{X}$ and $\mathcal{X}$, as only quadratic homogeneous vector fields $X \in \mathfrak{X}_2^H$ ($Y \in \mathcal{X}_2^H$, respect.) can present isolated (crossing, respect.) invariant cones, fulfilled of closed trajectories.

On the following we study the quadratic case.

Quadratic vector fields

The behavior of homogeneous vector fields is the same on all spheres. But this special property can not be extended for quadratic vector fields $\mathfrak{X}_2$.

Example

The quadratic system

$$(\dot{x}, \dot{y}, \dot{z}) = (-xz - yz - z^2 - z, -z^2, x^2 + xy + xz + yz + x)$$

is such that all the spheres are invariant by X and the equilibrium points are located at $p_\pm = (0, \pm\rho, 0)$ and at $\{x + y + 1 = z = 0\}$. So, in addition to $p_\pm$ we have two more when $\rho > 1/\sqrt{2}$ or one more when $\rho = 1/\sqrt{2}$.

We notice that in the above example the number of equilibrium points decrease from 4 to 2 when the plane $x + y + 1 = 0$ does not intersect the sphere of radius ρ.

We restrict our analysis to the unit sphere

$$\mathbb{S}_1^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

In this case we will show that, generically, any $X \in \mathfrak{X}_2$ writes in its canonical form as

$$\begin{aligned}
\dot{x} &= -a_1 y - a_2 z - a_4 xy - a_{10} y^2 - a_5 xz - (a_6 + a_7) yz - a_8 z^2, \\
\dot{y} &= a_1 x - a_3 z + a_4 x^2 + a_{10} xy + a_6 xz - a_{11} yz - a_9 z^2, \\
\dot{z} &= a_2 x + a_3 y + a_5 x^2 + a_7 xy + a_{11} y^2 + a_8 xz + a_9 yz.
\end{aligned} \tag{11}$$

The projected vector field Y has a weak focus on the origin when $a_4 = a_9$ and $a_2a_6 + a_6a_7 + 2a_2a_7 + {a_2}^2 + {a_7}^2 - {a_9}^2 > 0$.

We will add two extra conditions:

$$a_9 = 0 \text{ and } a_2 + a_7 = 1.$$

Then the projected vector field Y has a weak focus on the origin if, and only if, $a_4 = 0$ and $a_6 + 1 > 0$. Writing $w^2 = a_6 + 1$, with $w \neq 0$, we obtain

$$\begin{aligned}
\dot{x} &= -a_1y - (1 - a_7)z - a_4xy + a_1y^2 - a_5xz + (1 - a_7 - w^2)yz - a_8z^2, \\
\dot{y} &= a_1x + a_{11}z + a_4x^2 - a_1xy + (w^2 - 1)zx - a_{11}yz, \\
\dot{z} &= (1 - a_7)x - a_{11}y + a_5x^2 + a_7xy + a_{11}y^2 + a_8xz.
\end{aligned} \tag{12}$$

Then, after a reparametrization of the time, the projected system of (14) is

$$\begin{aligned}
\dot{u} &= -\frac{a_4}{w}u - v - \frac{a_1}{2}u^2 - \frac{a_5}{w}uv - \frac{a_1 + 2a_8}{2w^2}v^2 - \frac{a_4 w}{4}u^3 + \frac{2a_7 - w^2}{4}u^2 v \\
&\quad + \frac{a_4}{4w}uv^2 + \frac{w^2 + 2c_7 - 2}{4w^2}v^3 - \frac{a_1 w^2}{8}u^4 - \frac{a_{11} w}{4}u^3 v - \frac{a_{11}}{4w}uv^3 + \frac{a_1}{8w^2}v^4, \\
\dot{v} &= u + \frac{(2a_5 - a_{11})w}{2}u^2 + a_8 uv - \frac{a_{11}}{2w}v^2 - \frac{(2a_7 - 1)w^2}{4}u^3 - \frac{a_4 w}{2}u^2 v \\
&\quad - \frac{2w^2 + 2a_7 - 3}{4}uv^2 + \frac{w^3 a_{11}}{8}u^4 - \frac{w^2 a_1}{4}u^3 v - \frac{a_1}{4}uv^3 - \frac{a_{11}}{8w}v^4.
\end{aligned} \tag{13}$$

Theorem

The system

$$\begin{aligned}
\dot{x} &= -a_1 y - (1-a_7)z - a_4 xy + a_1 y^2 - a_5 xz + (1 - a_7 - w^2)yz - a_8 z^2,\\
\dot{y} &= a_1 x + a_{11} z + a_4 x^2 - a_1 xy + (w^2-1)zx - a_{11} yz,\\
\dot{z} &= (1-a_7)x - a_{11} y + a_5 x^2 + a_7 xy + a_{11} y^2 + a_8 xz,
\end{aligned} \tag{14}$$

has a center at the equilibrium point $(0,1,0)$ *if* $a_4 = 0$ *and one of the following conditions is satisfied:*

- a $w = 1,\ a_1 a_5 + a_8 a_{11} = 0;$
- b $a_1 = 0,\ \ a_8 = 0;$
- c $a_5 = 0,\ \ a_{11} = 0;$
- d $a_1 = a_8,\ \ a_5 = -a_{11};$
- e $w \neq 1,$
  $a_1 = \dfrac{w^2-1}{w^2+1} a_8,\ a_5 = \dfrac{w^2+1}{w^2-1} a_{11},\ a_7 = \dfrac{1}{w^2+1} - \dfrac{1}{(w^2+1)} a_8^2 - \dfrac{w^2+1}{(w^2-1)^2} a_{11}^2.$

Let $X = X(\mathrm{x}, \mathrm{a}) \in \mathfrak{X}_2$ given by (14) where $\mathrm{x} = (x, y, z)$ and $\mathrm{a} = (a_1, a_5, a_7, a_8, a_{11}, w)$. Denoting $\mathrm{a}+\varepsilon^{\pm} = (a_1+\varepsilon_1^{\pm}, \ldots, w+\varepsilon_6^{\pm})$ we consider the piecewise perturbation of X defined by

$$Y(\mathrm{x}, \varepsilon) = \begin{cases} X(\mathrm{x}; \mathrm{a} + \varepsilon^{+}), \ z \geq 0, \\ X(\mathrm{x}; \mathrm{a} + \varepsilon^{-}), \ z \leq 0, \end{cases} \tag{15}$$

and the projected vector field associeted is of the form

$$\mathcal{P}_Y(\mathrm{u}, \varepsilon) = \begin{cases} \mathcal{P}_X(\mathrm{u}; \mathrm{a} + \varepsilon^{+}), \ v \geq 0, \\ \mathcal{P}_X(\mathrm{u}; \mathrm{a} + \varepsilon^{-}), \ v \leq 0, \end{cases} \tag{16}$$

where $\mathrm{u} = (u, v)$ and $\mathcal{P}_X(\mathrm{u}, 0)$ is given by (13).

Theorem

Consider the system

$$\begin{aligned}
\dot{x} &= -\frac{4}{5}y - \frac{13}{8}z - \frac{5}{2}xz + \frac{4}{5}y^2 - \frac{59}{8}yz - z^2, \\
\dot{y} &= \frac{4}{5}x + 2z - \frac{4}{5}xy + 8xz - 2yz, \\
\dot{z} &= \frac{13}{8}x - 2y + \frac{5}{2}x^2 - \frac{5}{8}xy + xz + 2y^2.
\end{aligned} \tag{17}$$

- a $(0,1,0)$ *is a center.*
- b *There exists a smooth quadratic perturbation of* (17) *in* $\mathfrak{X}$ *such that at least* 3 *hyperbolic limit cycles of small amplitude bifurcate from the equilibrium point* $(0,1,0)$ *on* $\mathbb{S}^2_1$.
- c *There exists a piecewise quadratic perturbation of* (17) *in* $\mathcal{X}$ *such that at least* 10 *hyperbolic crossing limit cycles of small amplitude bifurcate from the equilibrium point* $(0,1,0)$ *on* $\mathbb{S}^2_1$.

Note that system (17) is obtained doing $a_1 = 4/5, a_4 = 0, a_5 = 5/2, a_7 = -5/8, a_8 = 1, a_{11} = 2$, and $w = 3$ in (14).

We consider the piecewise perturbation

$$(a_1, a_5, a_7, a_8, a_{11}, w) = (4/5 + \varepsilon_1^{\pm}, 5/2 + \varepsilon_2^{\pm}, -5/8 + \varepsilon_3^{\pm}, 1 + \varepsilon_4^{\pm}, 2 + \varepsilon_5^{\pm}, 3 + \varepsilon_6^{\pm})$$

in the projected system (13). We denote by $L_i(\varepsilon)$, with $\varepsilon = (\varepsilon_1^+, \dots, \varepsilon_6^+, \varepsilon_1^-, \dots, \varepsilon_6^-)$, the corresponding Lyapunov constants. Clearly, when $\varepsilon = 0$ the origin is a center and then $L_i(0) = 0$ for all i. We compute the Taylor series of the Lyapunov constants up to first-order with respect to ε, $L_i^{[1]}(\varepsilon)$, and we write

$$L_i(\varepsilon) = L_i^{[1]}(\varepsilon) + \mathcal{O}_2(\varepsilon).$$

As the matrix formed with the coefficients of

$$(L_2^{[1]}, \dots, L_{12}^{[1]})$$

with respect to ε has rank 9 so adding the trace parameter and using the Melnikov Theory, we can get 9 hyperbolic crossing limit cycles of small amplitude bifurcating from the origin. Adding the sliding parameter we get a pseudo-Hopf bifurcation and the proof follows.

Moltes gràcies!

¡Muchas gracias!

Thank you!

ICMC USP SÃO CARLOS 50 anos 1971 - 2021

FAPESP FUNDAÇÃO DE AMPARO À PESQUISA DO ESTADO DE SÃO PAULO