WORKSHOP ON DYNAMICAL SYSTEMS

Lleida, Thursday 11 – Friday 12 of January 2024

Recurrence of orbits in a 3D continuous PWL system

(joint work by Emilio Freire, E. Ponce & Javier Ros)

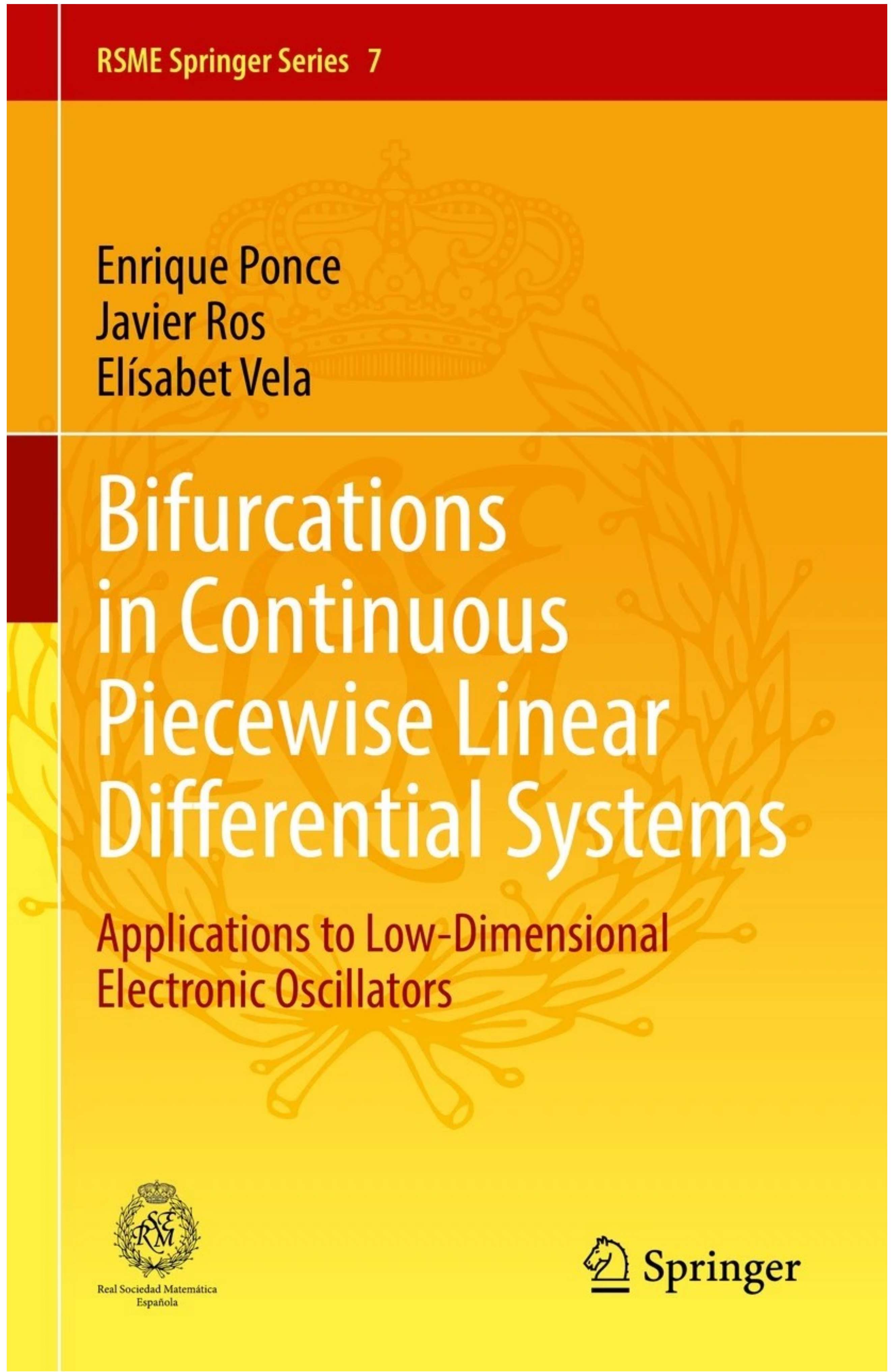
Felicitats: Per molts anys!

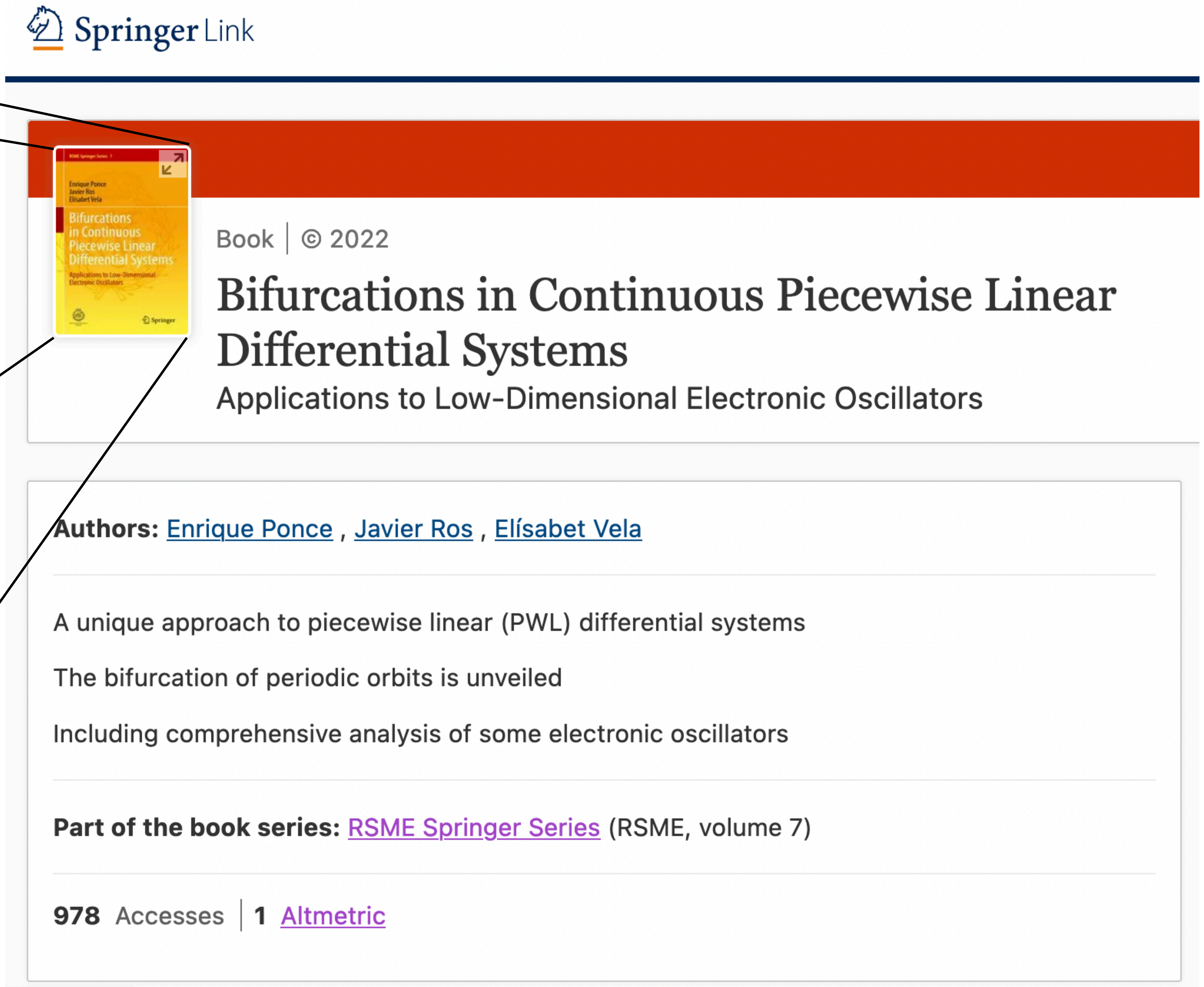
Unión Europea
Fondo Europeo de Desarrollo Regional
"Una manera de hacer Europa"

Junta de Andalucía
Consejería de Transformación Económica, Industria, Conocimiento y Universidades

Andalucía se mueve con Europa

First of all: we recommend a unavoidable reference for continuous piecewise linear systems





Abstract

We consider a specific piecewise linear differential system in 3D, with a vector field composed by two linear ones having continuity in the separation plane. One dynamics (the so referred as the left one) comes from a Hopf-zero singularity, while the other (the right one) is rather general. The system is relevant in the analysis of boundary equilibrium bifurcations.

We study the case of saddle-focus dynamics in the right zone, and show that the helicoidal return provided by the left zone can produce very interesting dynamics.

In particular, we show a recurrence property for the orbits that allows us to conjecture the existence of periodic orbits as well as non-periodic attractors.

The Sevillator model

We are interested in the dynamical behaviour of the continuous piecewise linear system given by the vector field

$$(X \leq 0) \begin{cases} \dot{X} &= -Y, \\ \dot{Y} &= X - Z, \\ \dot{Z} &= 1, \end{cases} \qquad (X > 0) \begin{cases} \dot{X} &= tX - Y, \\ \dot{Y} &= mX - Z, \\ \dot{Z} &= dX + 1. \end{cases}$$

Here the dot denotes derivatives with respect to the time variable τ, while t, m and d stand for the trace, the sum of second order minors and the determinant, respectively, of the matrix ruling the dynamics in the 'right' system.

The Sevillator model

We are interested in the dynamical behaviour of the continuous piecewise linear system given by the vector field

$$(X \leq 0) \begin{cases} \dot{X} &= -Y, \\ \dot{Y} &= X - Z, \\ \dot{Z} &= 1, \end{cases} \qquad (X > 0) \begin{cases} \dot{X} &= tX - Y, \\ \dot{Y} &= mX - Z, \\ \dot{Z} &= dX + 1. \end{cases}$$

Regarding the existence of periodic orbits for our system, we note first the following straightforward result.

Lemma If $d \geq 0$ then the system cannot have periodic orbits.

The Sevillator model

Hereafter, we assume $d < 0$.

$$(X \leq 0) \begin{cases} \dot{X} &= -Y, \\ \dot{Y} &= X - Z, \\ \dot{Z} &= 1, \end{cases} \qquad (X > 0) \begin{cases} \dot{X} &= tX - Y, \\ \dot{Y} &= mX - Z, \\ \dot{Z} &= dX + 1. \end{cases}$$

Thus, the system has one real equilibrium point at the right zone, namely

$$(\overline{X}, \overline{Y}, \overline{Z}) = (-1/d, -t/d, -m/d).$$

Periodic orbits can arise from a center at the focal plane

Under the assumption of complex eigenvalues for such equilibrium point, we can assure the existence of periodic orbits after some bifurcation of type *focus-center-limit cycle*, see Carmona *et al.* (2005).

In such a case, the equilibrium point of saddle-focus type becomes a saddle-center that leads to a bounded period annulus in the focal plane, with the biggest periodic orbit tangent to the separation plane $X = 0$. This critical situation appears when $m > 0$ and $m\,t - d = 0$ so that, if we consider t as being the bifurcation parameter then its critical value is $t_c = \frac{d}{m} < 0$.

Periodic orbits can arise from a center at the focal plane

Proposition Assume $0 < m < 1$ and $d < 0$. The system undergoes a focus-center-limit cycle bifurcation for $t = t_c < 0$; that is, from the linear center configuration that exists for $X > 0$ when $t = t_c$, one orbitally asymptotically stable limit cycle appears for $t > t_c$ and $t - t_c$ sufficiently small.

LIMIT CYCLE BIFURCATION IN 3D CONTINUOUS PIECEWISE LINEAR SYSTEMS WITH TWO ZONES. APPLICATION TO CHUA'S CIRCUIT

V. CARMONA, E. FREIRE, E. PONCE, J. ROS and F. TORRES

International Journal of Bifurcation and Chaos, Vol. 15, No. 10 (2005) 3153–3164

Periodic orbits can arise from a center at infinity

Proposition Assume $0 < m < 1$ and $d < 0$. The system undergoes a limit cycle bifurcation from infinity for $t = d$, so that one orbitally asymptotically stable limit cycle of great amplitude appears for $t < d$ and $d - t$ sufficiently small.

Bifurcations from a center at infinity in 3D piecewise linear systems with two zones

Emilio Freire, Manuel Ordóñez, Enrique Ponce * Physica D 402 (2020) 132280

We dared to state a seemingly natural conjecture…

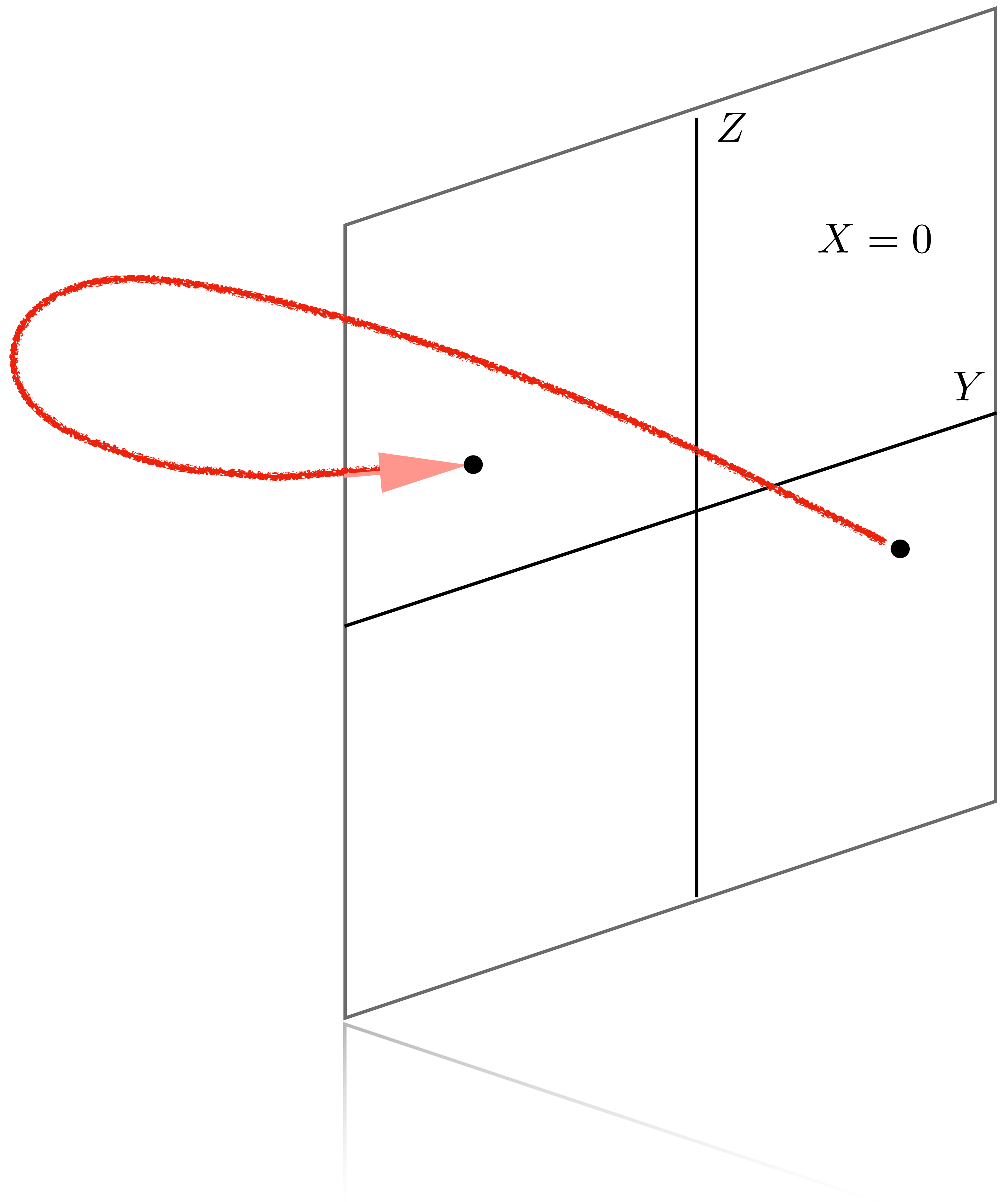
Conjecture The system with $0 < m < 1$ and $d < 0$ has one stable limit cycle for any value of t with $t_c < t < d$.

Limit Cycle Bifurcation from a Persistent Center at Infinity in 3D Piecewise Linear Systems with Two Zones

Emilio Freire, Manuel Ordóñez, and Enrique Ponce

A. Colombo et al. (eds.), *Extended Abstracts Spring 2016*, Trends in Mathematics 8, DOI 10.1007/978-3-319-55642-0_10

The left dynamics



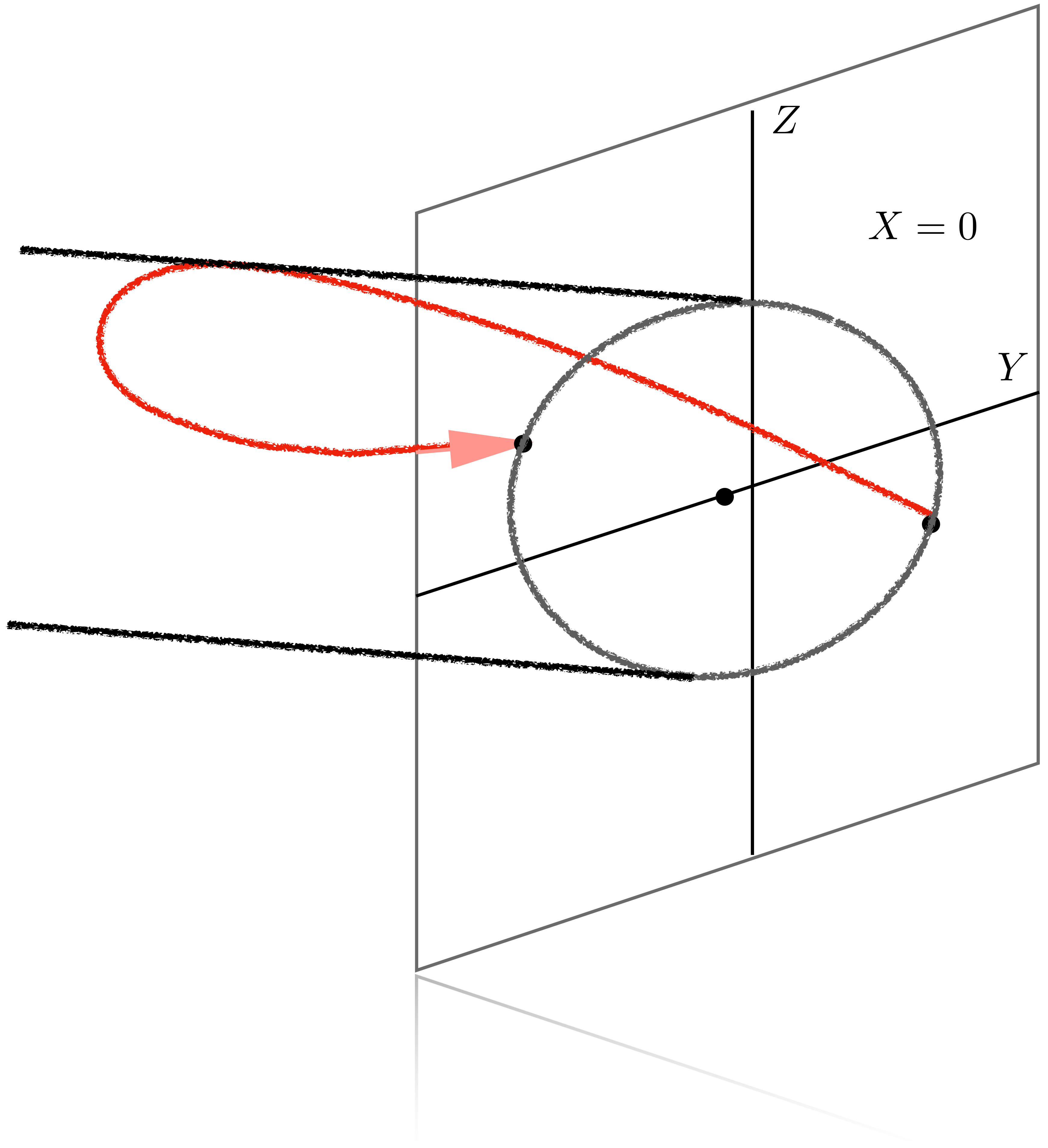
Taking as initial point $(0, Y_0, Z_0)$ with $Y_0 > 0$, by direct integration of the left vector field, we can write

$$\begin{aligned} X(\tau) &= -(1+Y_0)\sin\tau + Z_0(1-\cos\tau) + \tau, \\ Y(\tau) &= (1+Y_0)\cos\tau - Z_0\sin\tau - 1, \\ Z(\tau) &= Z_0 + \tau, \end{aligned}$$

and then it is easy to see the following result.

Lemma Any orbit of the system starting at $(0, Y_0, Z_0)$ with $Y_0 > 0$ enters the half-space $X < 0$ and returns to the plane $X = 0$ after a time $0 < \tau < 2\pi$, so that $X(\tau) = 0$.

The left dynamics



The left system has the first integral

$$H(X,Y,Z) = (X - Z)^2 + (Y + 1)^2,$$

so that the cylinders $H(X,Y,Z) = k$, which share as their common axis the straight line $X = Z, Y = -1$, are invariant for $X < 0$.

The right dynamics

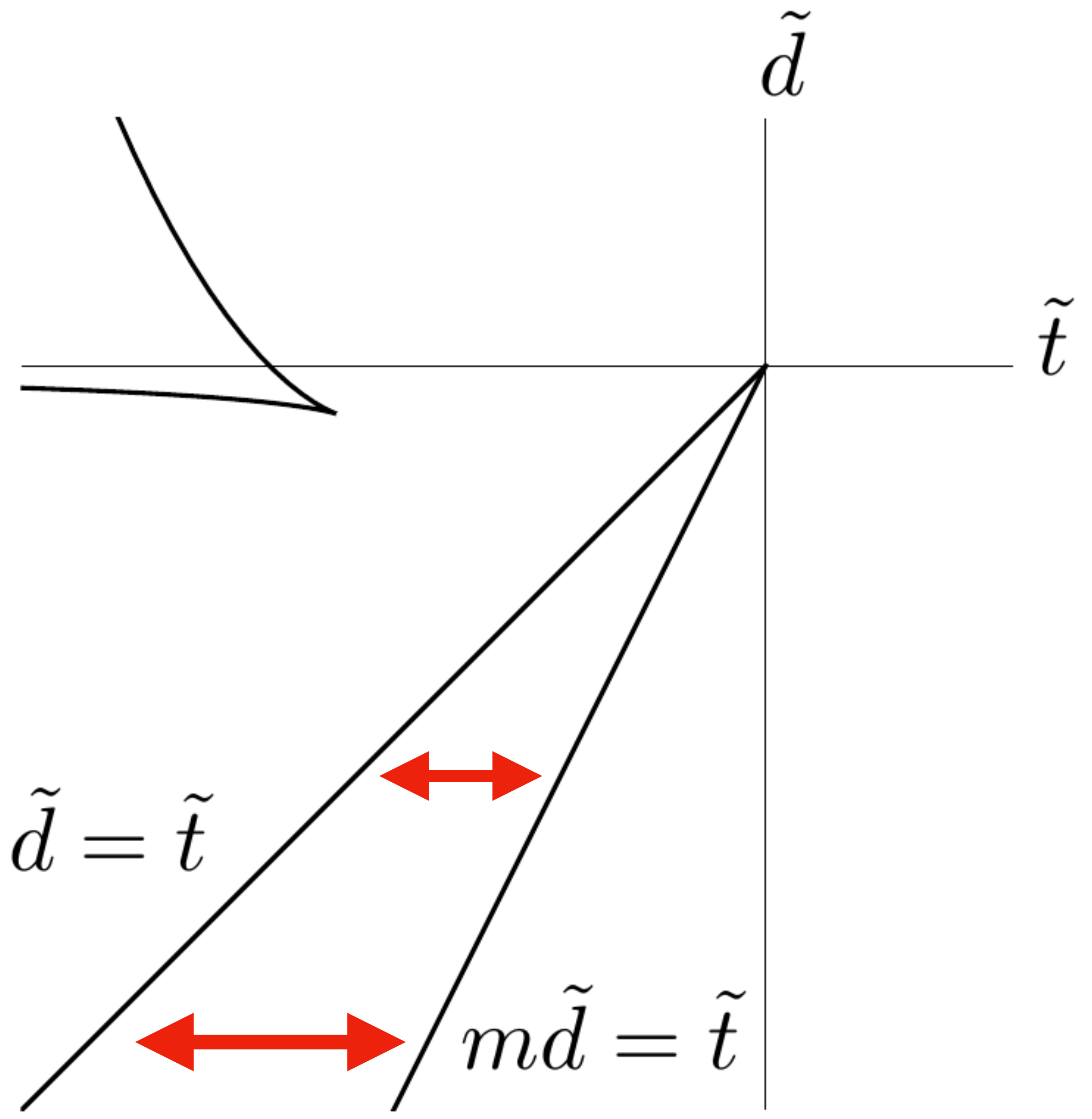
Proposition The right vector field with $0 < m < 1$ and $d < 0$ has for $t_c < t < d$ a real eigenvalue $\lambda < 0$ and a complex eigenvalue pair $\sigma \pm \mathrm{i}\,\omega$, with $\sigma > 0$ and $0 < \omega < 1$.

We start from the characteristic polynomial associated to its linear matrix, namely

$$\xi^3 - t\xi^2 + m\xi - d = 0,$$

by rescaling the roots and coefficients so that $\xi = m^{1/2}\mu$, $t = m^{1/2}\widetilde{t}$, $d = m^{3/2}\widetilde{d}$, getting the normalized polynomial

$$\mu^3 - \widetilde{t}\mu^2 + \mu - \widetilde{d} = 0.$$



In the plane $(\tilde{t}, \tilde{d})$, for $\tilde{t} < 0$, the cuspidal region corresponding to three real roots is far from the sectorial region of interest $\tilde{d} < \tilde{t} < m\tilde{d}$.

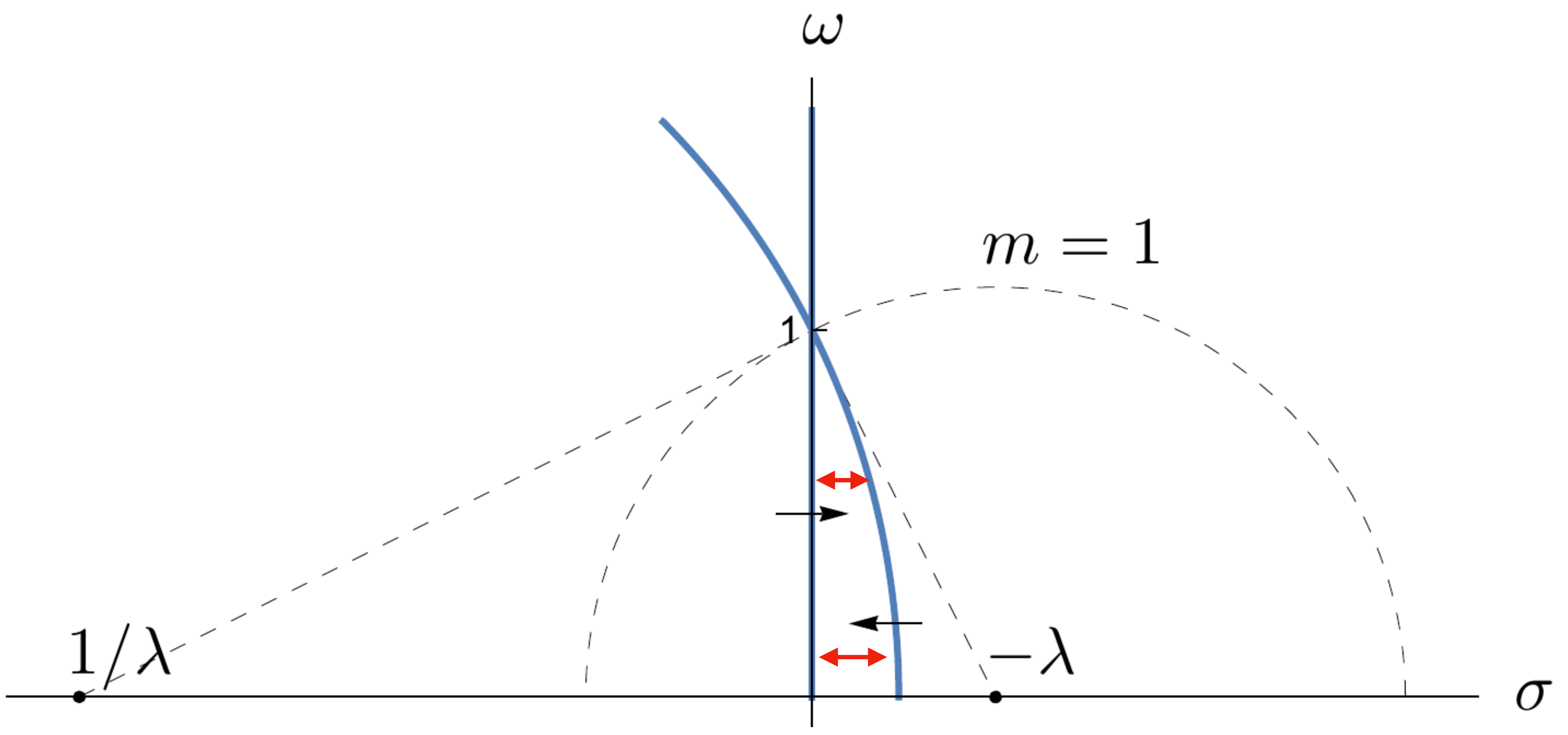
The preliminary bifurcation set

Once we know for sure the structure of eigenvalues, we can write for the original parameters the equalities

$$
\begin{aligned}
t &= \lambda + 2\sigma, \\
m &= 2\lambda\sigma + \sigma^2 + \omega^2, \\
d &= \lambda(\sigma^2 + \omega^2),
\end{aligned}
$$

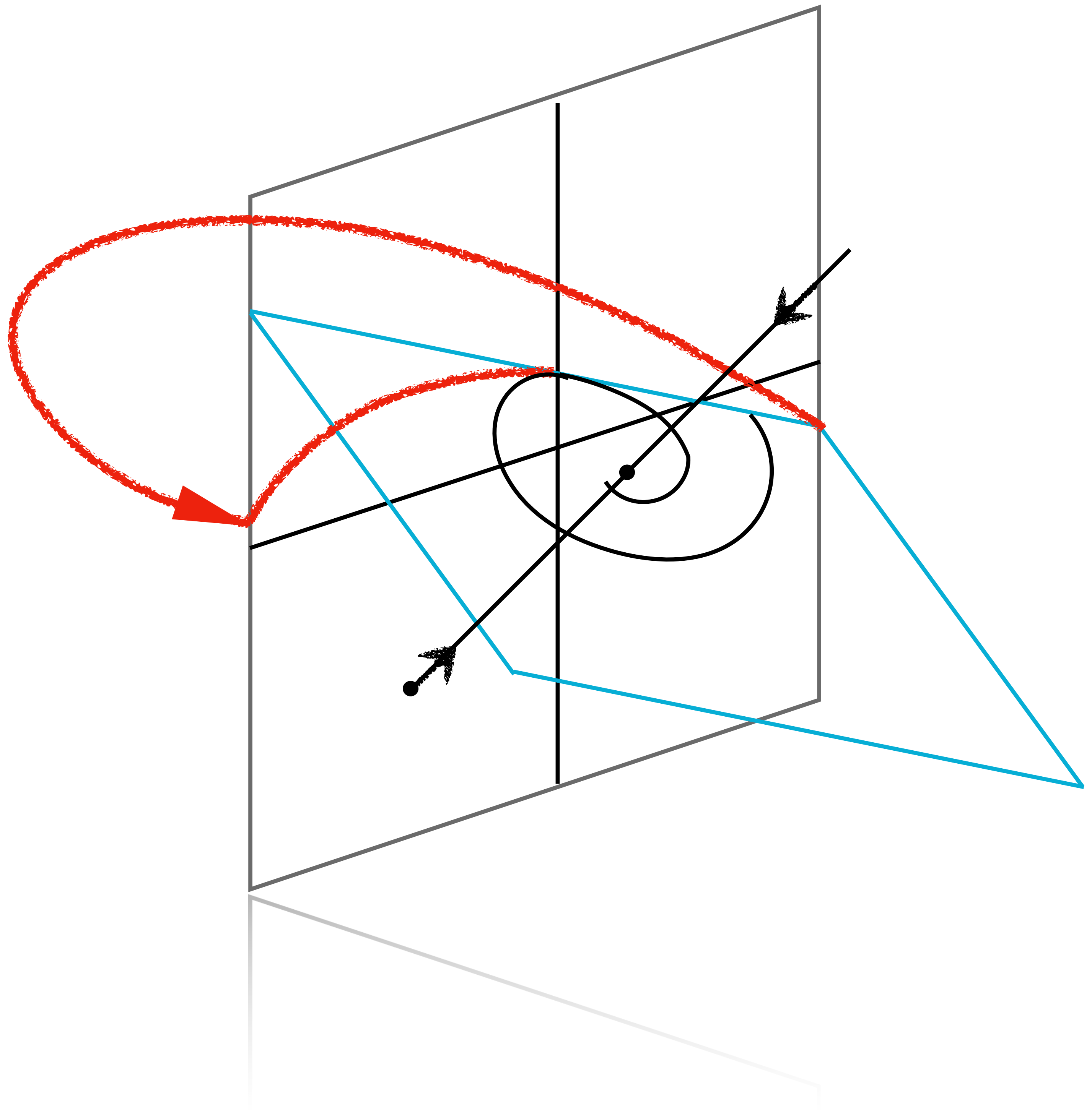
so that it is clear that $\lambda < 0$, and a simple computation gives

$$m\,t - d = 2\sigma[(\lambda+\sigma)^2 + \omega^2].$$



Preliminary bifurcation set in the plane (σ, ω) for a fixed value of λ (the case $\lambda = -1/2$ is drawn). The black arrows indicate the known bifurcations of periodic orbits, supporting our Conjecture. We are interested in the region with $\sigma > 0$ and $t < d$.

The geometry of orbits



Proposition Under the hypotheses $0 < m < 1$, $d < 0$ and $t_c < t < d$, if $\lambda < 0$ is the real eigenvalue of the right system, then the focal plane

$$\Pi(X, Y, Z) := \lambda^2 X - \lambda Y + Z + 1/\lambda = 0$$

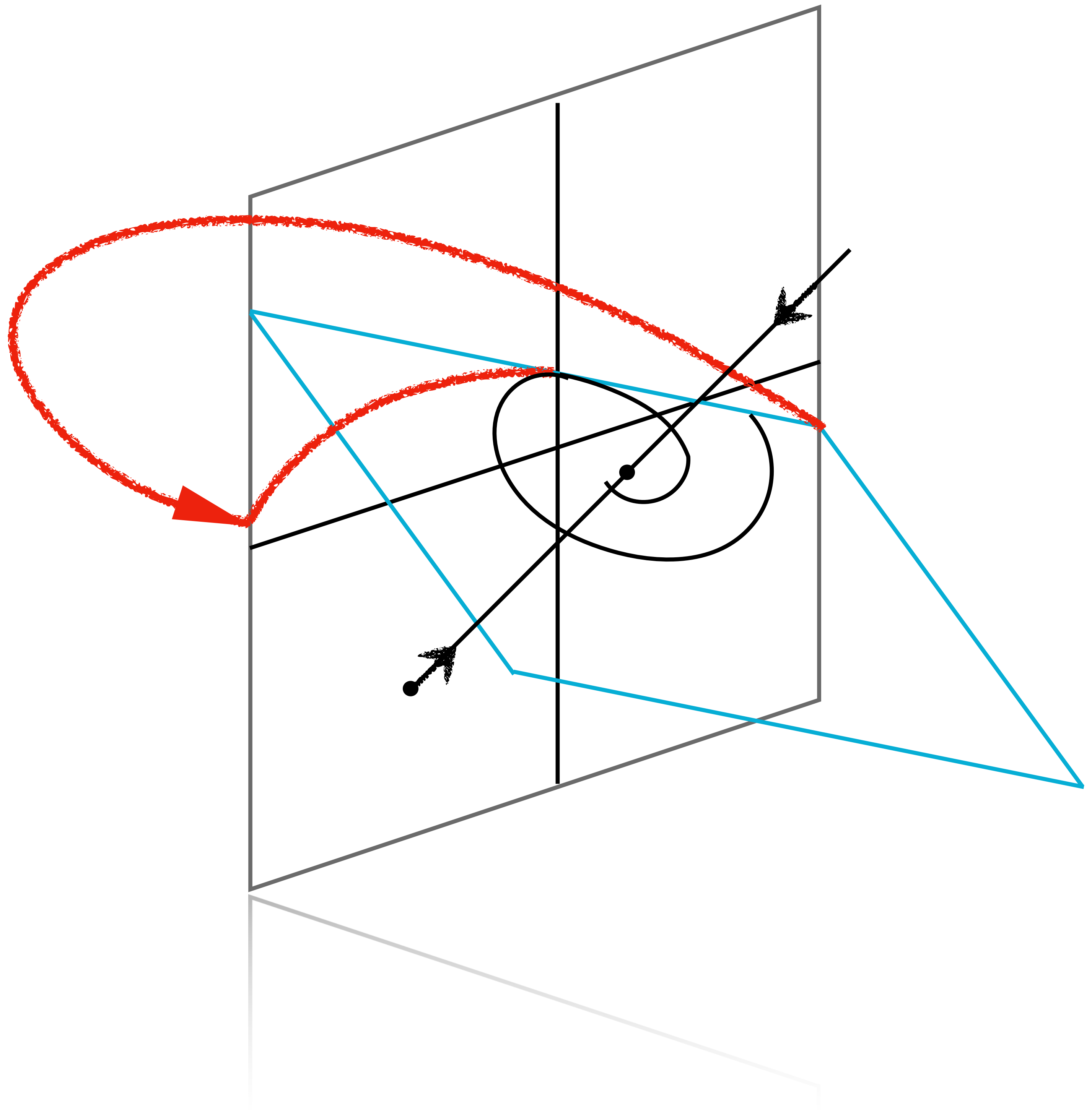
is invariant for its dynamics. Furthermore, any orbit of the left system with initial point in the planar set

$$\{(X, Y, Z) : X = 0, Y > 0, Z \leq \lambda Y - 1/\lambda\}$$

will return after a time $0 < \tau < 2\pi$ to a point of the planar set

$$\{(X, Y, Z) : X = 0, Y < 0, Z < \lambda Y - 1/\lambda\}.$$

The geometry of orbits



Proposition Under the hypotheses $0 < m < 1$, $d < 0$ and $t_c < t < d$, if $\lambda < 0$ is the real eigenvalue of the right system, then the focal plane

$$\Pi(X, Y, Z) := \lambda^2 X - \lambda Y + Z + 1/\lambda = 0$$

is invariant for its dynamics. Furthermore, any orbit of the left system with initial point in the planar set

$$\{(X, Y, Z) : X = 0, Y > 0, Z \leq \lambda Y - 1/\lambda\}$$

will return after a time $0 < \tau < 2\pi$ to a point of the planar set

$$\{(X, Y, Z) : X = 0, Y < 0, Z < \lambda Y - 1/\lambda\}.$$

Proof We first note that

$$(\lambda^2, -\lambda, 1)\begin{pmatrix} \lambda + 2\sigma & -1 & 0 \\ 2\lambda\sigma + \sigma^2 + \omega^2 & 0 & -1 \\ \lambda(\sigma^2 + \omega^2) & 0 & 0 \end{pmatrix} = (\lambda^3, -\lambda^2, \lambda),$$

so that $(\lambda^2, -\lambda, 1)$ is a left eigenvector associated to the eigenvalue λ for the matrix of right system. Accordingly, we have for such a vector field

$$\frac{d}{d\tau}\Pi(X, Y, Z) = \lambda^2 \dot{X} - \lambda\dot{Y} + \dot{Z} = \lambda^3 X - \lambda^2 Y + \lambda Z + 1,$$

that is,

$$\frac{d}{d\tau}\Pi(X, Y, Z) = \lambda\Pi(X, Y, Z).$$

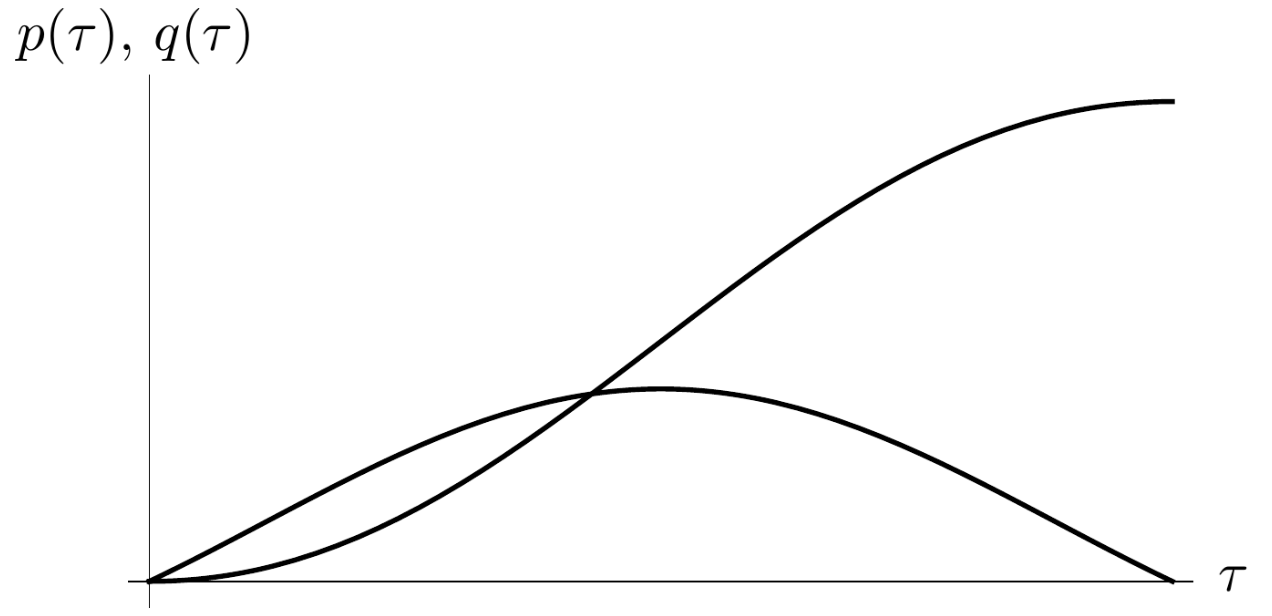
Hence, the invariance of the plane $\Pi(X, Y, Z) = 0$ under the flow of the right system follows.

Proof (Cont'd) We consider now the points in the half straight line with $Y > 0$ where this focal plane intersects the separation plane $X = 0$. For any of such points $(0, Y_0, \lambda Y_0 - 1/\lambda)$ we compute the corresponding return point $(0, Y_1, Z_1)$, namely

$$\begin{aligned} 0 &= -(1+Y_0)\sin\tau + (\lambda Y_0 - 1/\lambda)(1-\cos\tau) + \tau, \\ Y_1 &= (1+Y_0)\cos\tau - (\lambda Y_0 - 1/\lambda)\sin\tau - 1, \\ Z_1 &= \lambda Y_0 - 1/\lambda + \tau, \end{aligned}$$

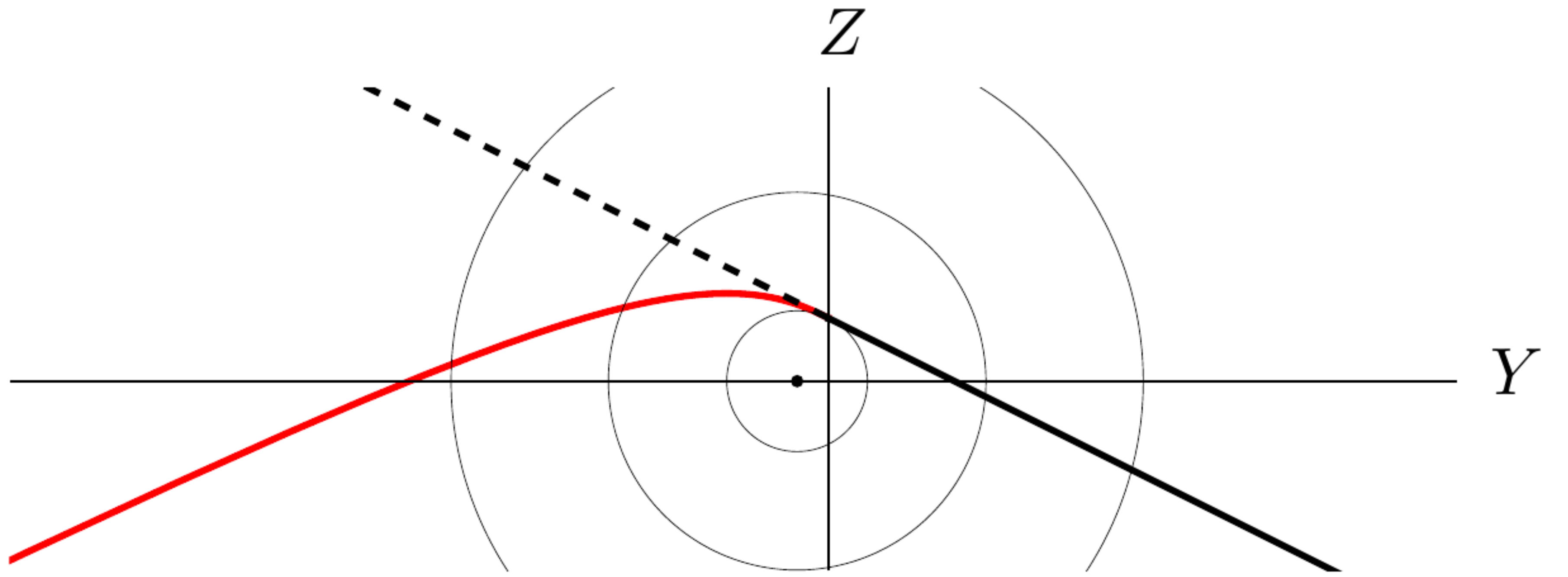
and solving the first equation for Y_0, we arrive to a parameterization of Y_0 in terms of the return time τ, namely

$$Y_0(\tau) = -\frac{p(\tau)}{\lambda q(\tau)} = -\left(\frac{1}{\lambda}\right)\frac{1-\cos\tau-\lambda(\tau-\sin\tau)}{\sin\tau-\lambda(1-\cos\tau)}.$$



Here, $0 < \tau < \tau_{\max}$, where $\tau_{\max} = 2\pi + 2\arctan(1/\lambda) \in (\pi, 2\pi)$ is the first positive zero of $q(\tau)$.

Proof (Cont'd) We have



$$\begin{aligned} Y_1(\tau) &= Y_0(\tau)q'(\tau) + q(\tau)/\lambda \\ Z_1(\tau) &= \lambda Y_0(\tau) - 1/\lambda + \tau, \end{aligned}$$

so that, for $0 < \tau < \tau_{\max}$, after some computations, we get

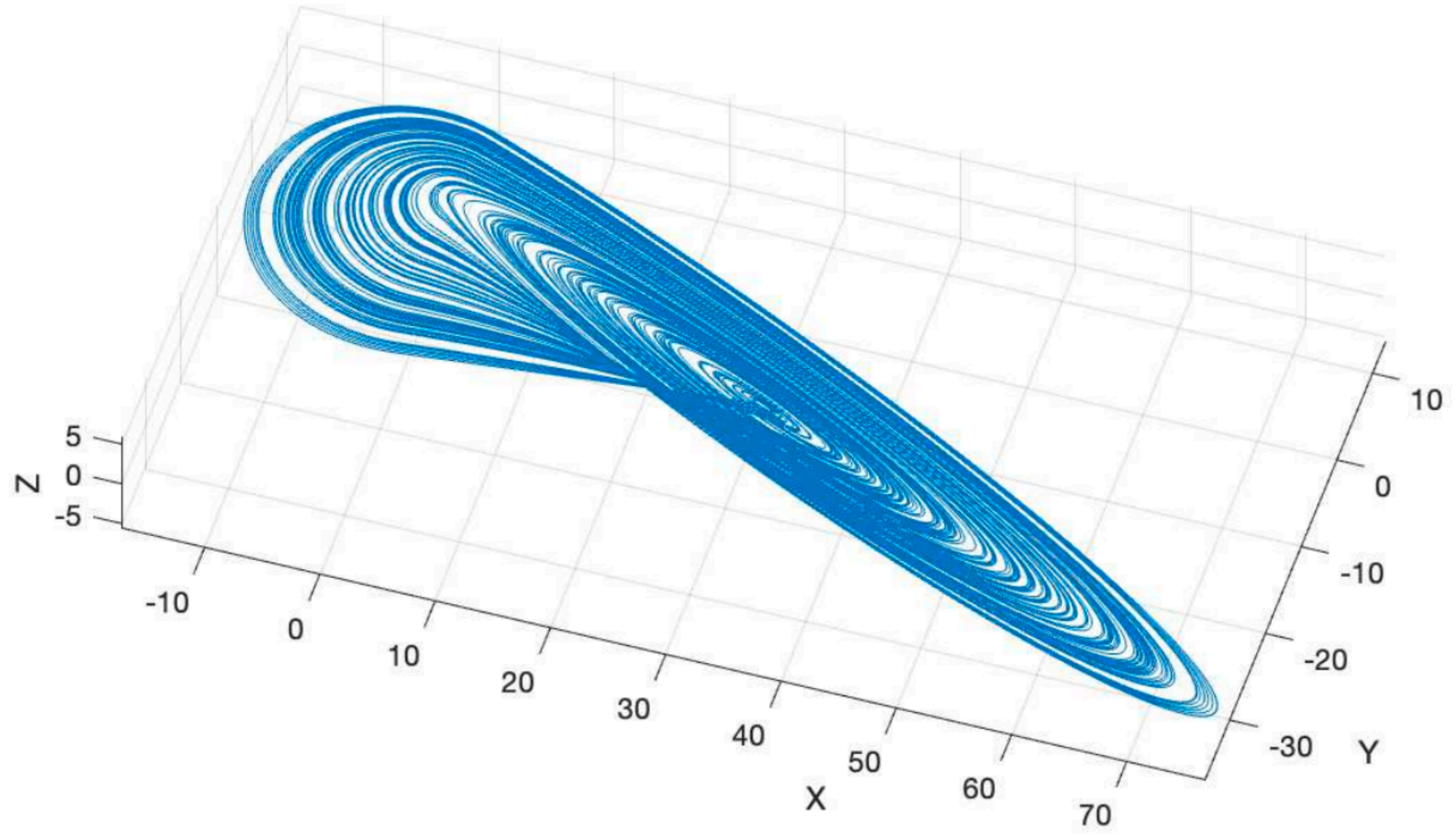
$$\lim_{\tau\to\tau_{\max}} \frac{Z_1(\tau)}{Y_1(\tau)} = \lim_{\tau\to\tau_{\max}} \frac{Z_1'(\tau)}{Y_1'(\tau)} = \lim_{\tau\to\tau_{\max}} \frac{\lambda q'(\tau)}{1-\lambda q(\tau)} = -\lambda,$$

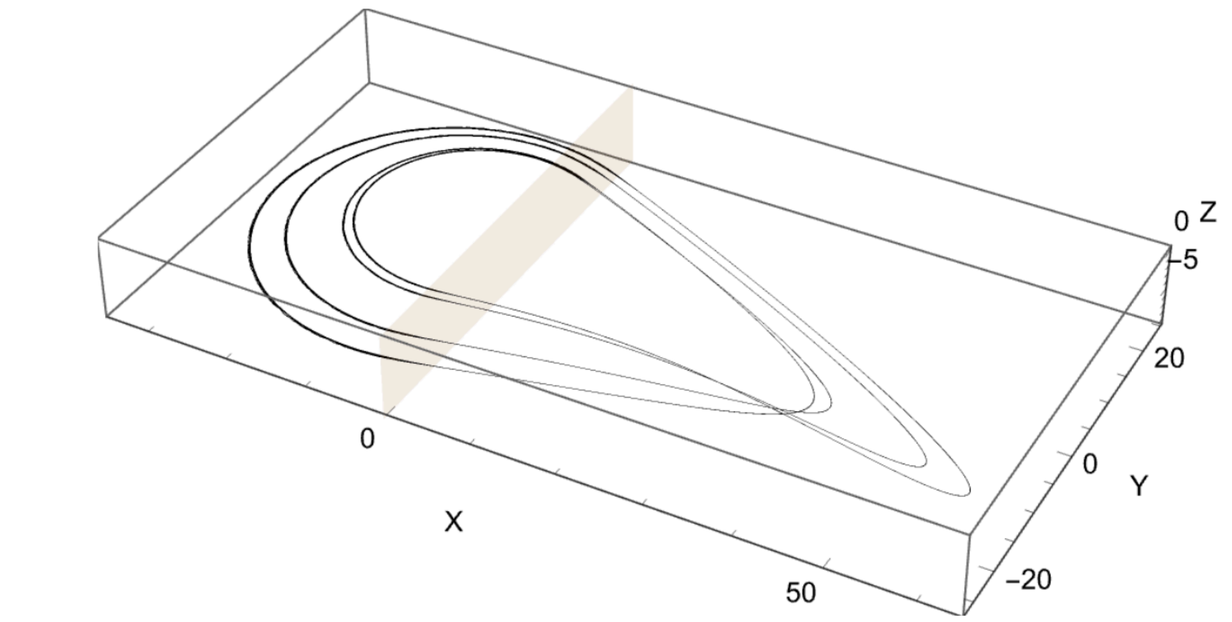
and

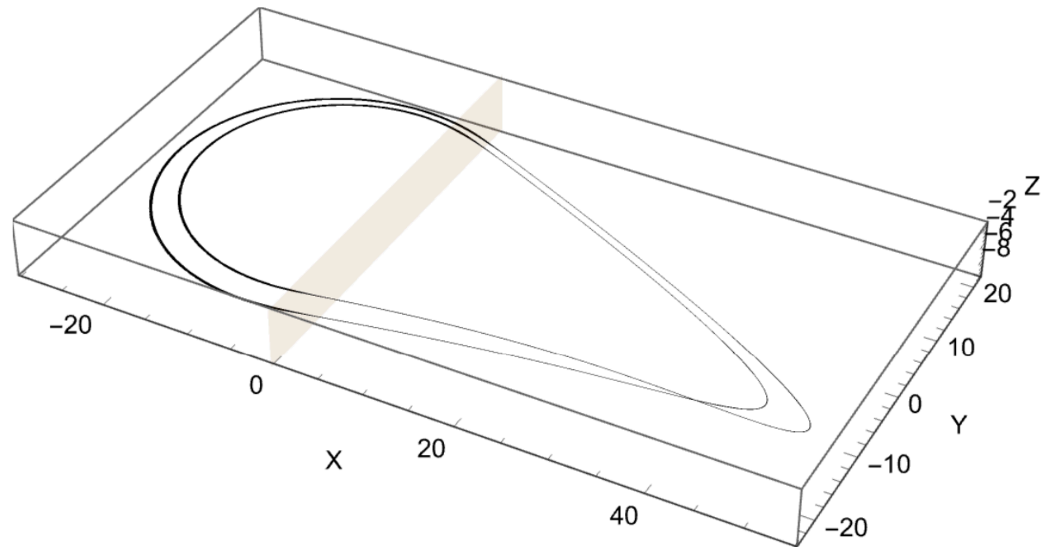
$$\frac{d^2 Z_1}{dY_1^2}(\tau) = \frac{Z_1''Y_1' - Z_1'Y_1''}{(Y_1')^3}(\tau) = -\frac{\lambda^2(1+\lambda^2)q(\tau)^3}{p(\tau)\,(1-\lambda q(\tau))^3} < 0.$$

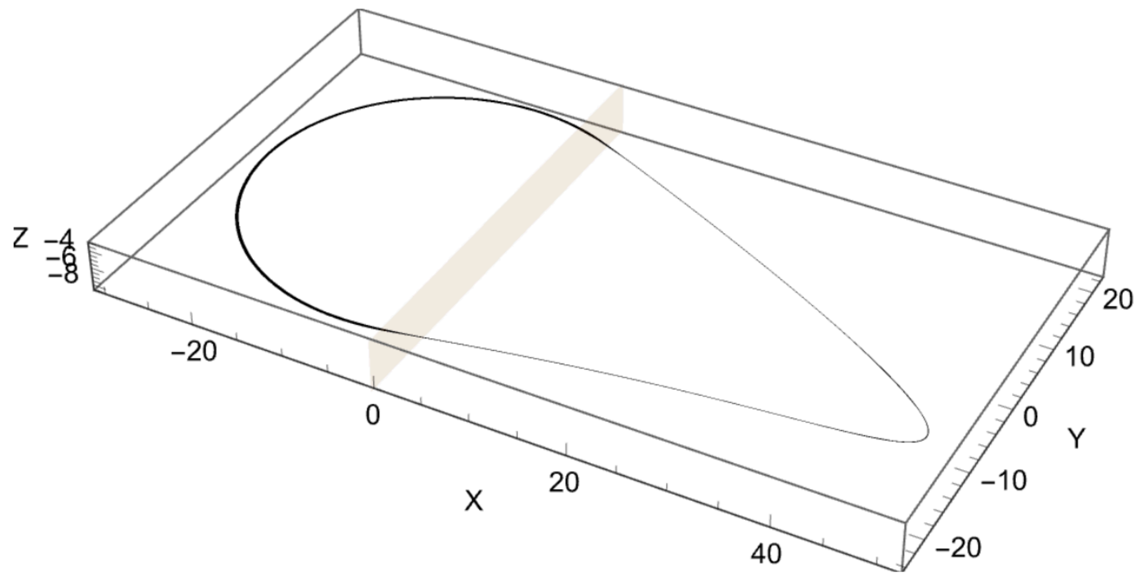
The conclusion is now a direct consequence of having for all $\tau \in (0, \tau_{\max})$ a negative second derivative.

Finding chaos when it was not looked for

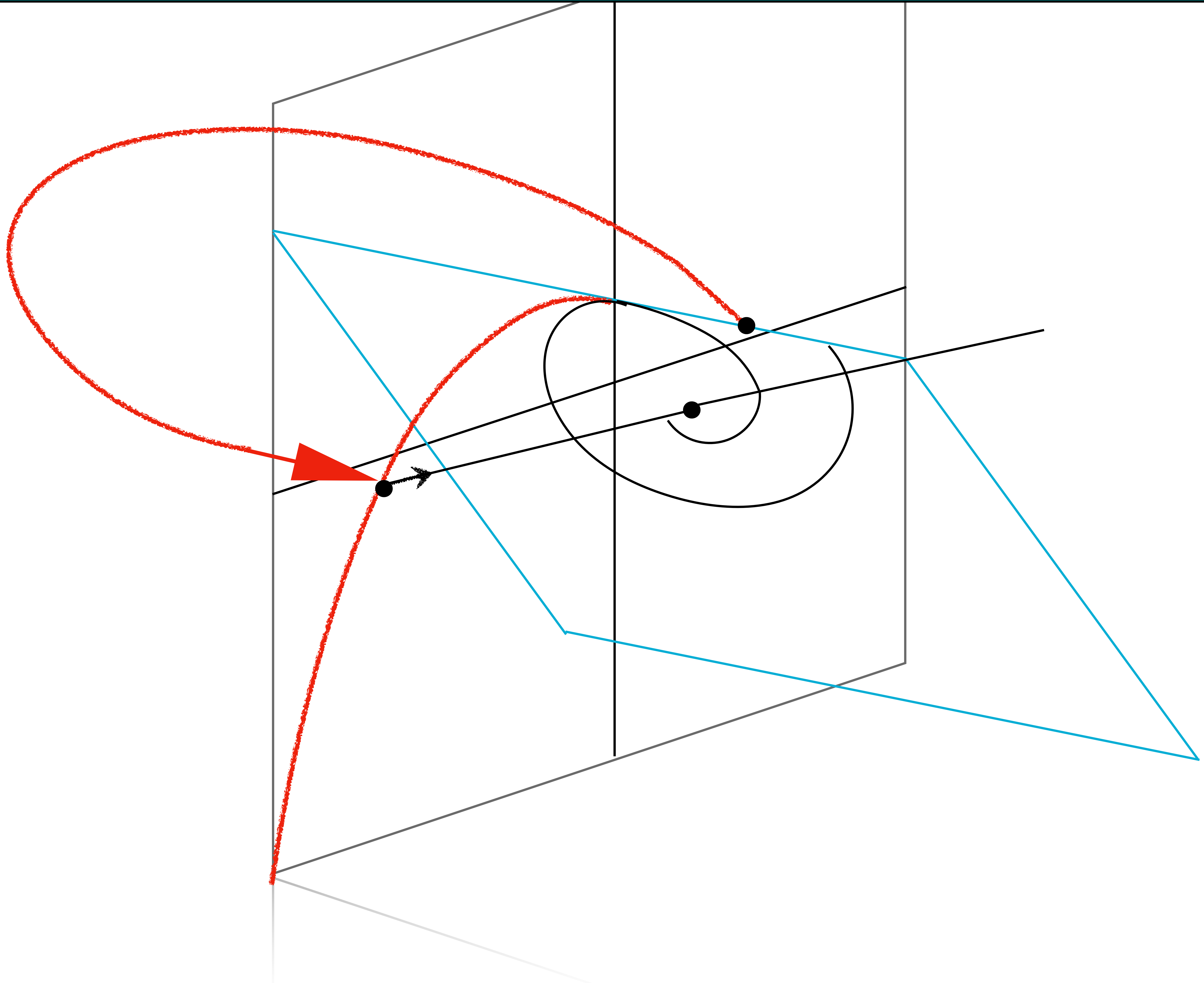




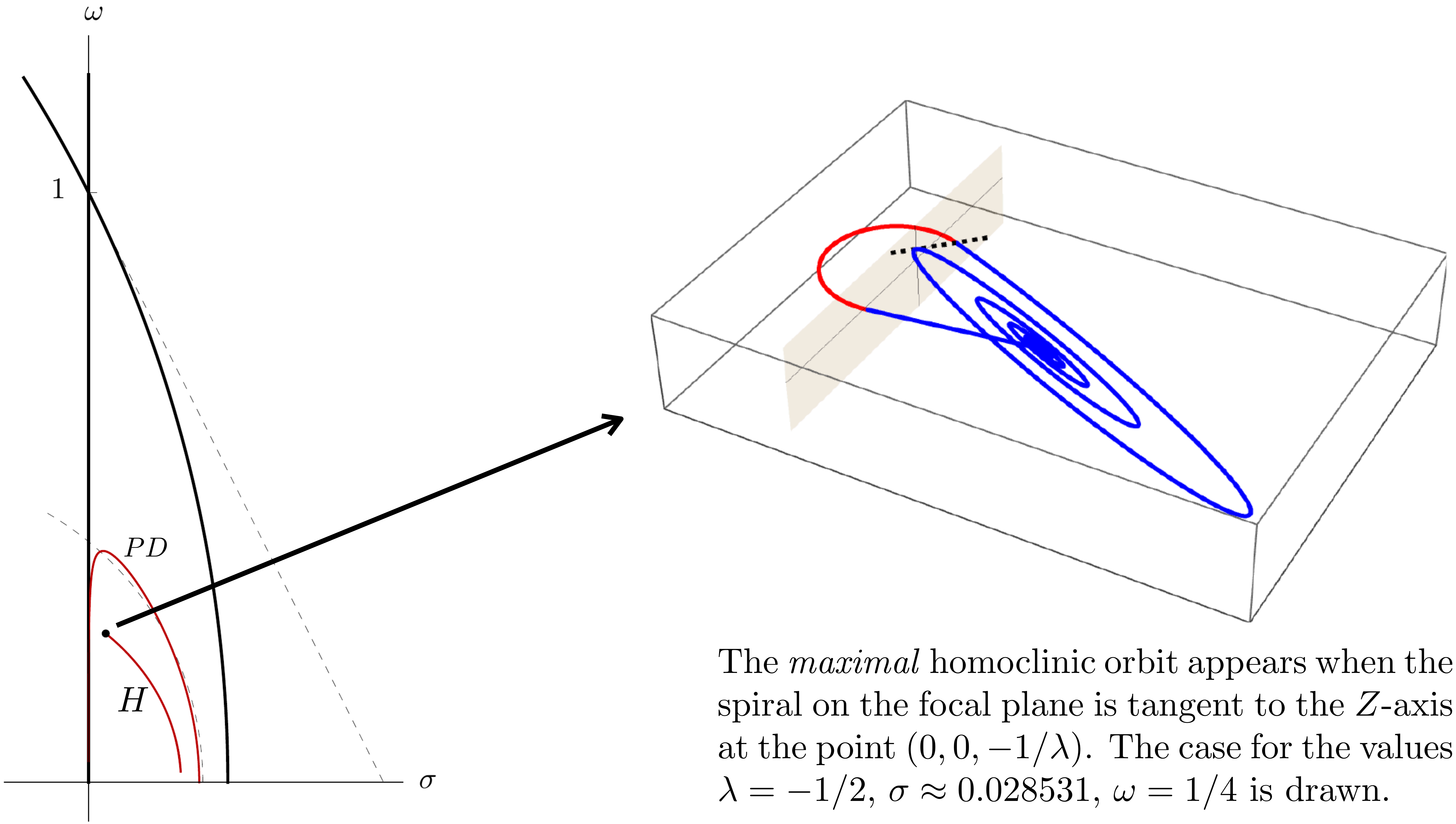




Finding chaos when not looked for: homoclinic orbits appear!



A more realistic bifurcation set



The *maximal* homoclinic orbit appears when the spiral on the focal plane is tangent to the Z-axis at the point $(0, 0, -1/\lambda)$. The case for the values $\lambda = -1/2$, $\sigma \approx 0.028531$, $\omega = 1/4$ is drawn.

Conclusions

- In the last decades, many authors have worked in looking for chaotic systems. Here, unexpectedly, we have found Shilnikov chaos in one of the simplest piecewise linear models, when trying to check a natural conjecture on existence of stable limit cycles.
- Accordingly, the mentioned conjecture has been shown to be not true. Anyway, the intrinsical interest of the model deserves a much deeper analysis regarding its usefulness in the characterization of boundary equilibrium bifurcations for 3D piecewise linear systems.
- In particular, this partial study paves the way for characterizing the direct transition to chaos in such bifurcations.