Workshop on Dynamical Systems

Lleida 11-12 January, 2024

Normal Forms for Discontinuous Systems

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(Joint work with Emilio Freire, Enrique Ponce and Marina Esteban)

INTRODUCTION

The method of normal forms for smooth vector fields with an equilibrium point is very well known.

To determine normal forms for piecewise smooth vector fields is a different issue. We have different vector fields separated by a discontinuar line, so the change of variables depend on the zone where the change acts. Piecewise smooth vector fields can have at the separation boundary pseudo-equilibrium points where the vector fields do not vanish but they are true equilibrium for the whole vector field.

Moreover, to guarantee the topological equivalence of the initial vector field ant the simplified one we have to use only change of variables which make invariant every point of the discontinuity line.

The main aim of this talk is to propose a method to obtain normal forms for planar piecewise smooth vector fields with a discontinuity line in a neighborhood of a pseudo-focus.

References in the past century

- N.N. Bautin and E.A. Leontovich, Methods and Techniques of Qualitative Study of Dynamic Systems in a Plane, Nauka, Moscow, 1976 (In Russian).
- N. A. Gubar, *Bifurcations in the Vicinity of a Fused Focus,* Journal Applied Mathematics and Mechanics, **35** (1971), 890–895.
- B. N. Skriabin, *The formation of a Limit Cycle at a Merged Focus,* Journal Applied Mathematics and Mechanics, **42** (1978), 952–955.
- A.F. Filippov, Differential equations with discontinuous righthand sides, Kluwer Academic Publishers Group, Dordrecht, 1988.

References

- B. Coll, A. Gasull and R. Prohens, *Degenerate Hopf Bifurcations in Discontinuous Planar Systems,* Journal of Mathematical Analysis and Applications, **253** (2001), 671–690.
- F. Liang and M. Han, *Degenerate Hopf Bifurcations in Nonsmooth Planar Systems,* International Journal of Bifurcations and Chaos, **22** (2012), 1250057.
- D.C. Braga, A. Fernandes da Fonseca, L.F. Gonçalves and L.F. Mello, *Lyapunov coefficients for an invisible fold-fold singularity in planar piecewise Hamiltonian systems,* Journal of Mathematical Analysis and Applications, **484**(2020), 123692.
- D.C. Braga, A. Fernandes da Fonseca, L.F. Gonçalves and L.F. Mello, *Limit cycles bifurcating from an invisible fold-fold in planar piecewise hamiltonian systems,* Journal of Dynamical and Control Systems, **27**(2021), 179–204.
- D.D. Novaes and L. Silva, *Lyapunov coefficients for monodromic tangential singularities in Filippov vector fields,* Journal of Differential Equations, **300**(2021), 565–596.

Setting of the problem

We consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{F}^-(\mathbf{x}) = (F_1^-(\mathbf{x}), F_2^-(\mathbf{x}))^\top, & \text{if } x \leqslant 0 \\ \mathbf{F}^+(\mathbf{x}) = (F_1^+(\mathbf{x}), F_2^+(\mathbf{x}))^\top, & \text{if } x \geqslant 0 \end{cases} \tag{1}$$

Here, $\mathbf{x} = (x, y)$

$x < 0$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{F}^-(\mathbf{x}) = \begin{pmatrix} F_1^-(x,y) \\ F_2^-(x,y) \end{pmatrix}$$

$x > 0$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{F}^+(\mathbf{x}) = \begin{pmatrix} F_1^+(x,y) \\ F_2^+(x,y) \end{pmatrix}$$

Setting of the problem

Main hypotheses:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{F}^-(\mathbf{x}) = \begin{pmatrix} F_1^-(x,y) \\ F_2^-(x,y) \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{F}^+(\mathbf{x}) = \begin{pmatrix} F_1^+(x,y) \\ F_2^+(x,y) \end{pmatrix}$$

The origin is a contact point from both sides

$$(\mathbf{H1}) \quad F_1^-(0,0) = F_1^+(0,0) = 0$$

with invisible quadratic tangencies

$$(\mathbf{H2}) \quad \frac{\partial F_1^-(0,0)}{\partial y} F_2^-(0,0) > 0, \qquad \frac{\partial F_1^+(0,0)}{\partial y} F_2^+(0,0) < 0$$

and monodromic

$$(\mathbf{H3}) \quad F_2^-(0,0) F_2^+(0,0) < 0$$

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Main hypotheses:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{F}^-(\mathbf{x}) = \begin{pmatrix} F_1^-(x,y) \\ F_2^-(x,y) \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{F}^+(\mathbf{x}) = \begin{pmatrix} F_1^+(x,y) \\ F_2^+(x,y) \end{pmatrix}$$

$$(\mathbf{H1}) \quad F_1^-(0,0) = F_1^+(0,0) = 0$$

$$(\mathbf{H2}) \quad \frac{\partial F_1^-(0,0)}{\partial y} F_2^-(0,0) > 0, \qquad \frac{\partial F_1^+(0,0)}{\partial y} F_2^+(0,0) < 0$$

$$(\mathbf{H3}) \quad F_2^-(0,0) F_2^+(0,0) < 0$$

The origin is a pseudo-focus point of quadratic type!

Normalized initial system

(after a rescaling in time and in x for each side and assuming positive rotation sense)

Proposition 1. *System (1) under hypotheses (H1)-(H3) can be written in the form*

$$\dot{\mathbf{x}} = \begin{pmatrix} F_1^-(\mathbf{x}) \\ F_2^-(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} a_{10}^- x - y + \sum_{p+q \geqslant 2} a_{pq}^- x^p y^q \\ -1 + b_{10}^- x + b_{01}^- y + \sum_{p+q \geqslant 2} b_{pq}^- x^p y^q \end{pmatrix} \quad \text{if } x \leqslant 0, \tag{3}$$

and

$$\dot{\mathbf{x}} = \begin{pmatrix} F_1^+(\mathbf{x}) \\ F_2^+(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} a_{10}^+ x - y + \sum_{p+q \geqslant 2} a_{pq}^+ x^p y^q \\ 1 + b_{10}^+ x + b_{01}^+ y + \sum_{p+q \geqslant 2} b_{pq}^+ x^p y^q \end{pmatrix} \quad \text{if } x \geqslant 0, \tag{4}$$

where the dot represents derivatives with respect to a new time τ.

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and

$$\dot{\mathbf{x}} = \begin{pmatrix} F_1^+(\mathbf{x}) \\ F_2^+(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} a_{10}^+ x - y + \sum\limits_{p+q\geqslant 2} a_{pq}^+ x^p y^q \\ 1 + b_{10}^+ x + b_{01}^+ y + \sum\limits_{p+q\geqslant 2} b_{pq}^+ x^p y^q \end{pmatrix} \quad \text{if } x \geqslant 0, \tag{4}$$

where the dot represents derivatives with respect to a new time τ.

Our goal is to get a locally topologically equivalent system as simple as possible: a normal form

Some remarks

- We must work by doing transformations on each side that preserve the points at the discontinuity line, in order to not destroy possible closed orbits.
- The left system becomes the right one, after the change of variables and parameters

$$x \to -x, \quad y \to y, \quad \tau \to -\tau, \quad a_{pq}^- \to (-1)^p a_{pq}^+, \quad b_{pq}^- \to (-1)^{p+1} b_{pq}^+$$

- Therefore, for the analysis, we only need to pay attention to one of the two sub-systems. For definiteness, we will work with the right one.

Quasi-homogeneus polynomial decomposition

(Key idea for controlling the influence of near-identity transformations)

Definitions

(a) A scalar polynomial function $f(\mathbf{x})$ is *quasi-homogeneous* of type (p_1, p_2) and order k if all its monomials $x^{a_1}y^{a_2}$ satisfy $a_1p_1 + a_2p_2 = k$.

(b) A polynomial vector field $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}))^\top$ is *quasi-homogeneous* of type (p_1, p_2) and degree k if both components are quasi-homogeneous functions of type (p_1, p_2), being the first component F_1 of order $k + p_1$ while the second component F_2 is of order $k + p_2$.

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Example for type (2,1) quasi-homogeneity (the exponent of x is multiplied by 2, the one of y by 1):

$$\begin{pmatrix} a_{03}y^3 + a_{11}xy \\ b_{02}y^2 + b_{10}x \end{pmatrix}$$

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Example for type (2,1) quasi-homogeneity (the exponent of x is multiplied by 2, the one of y by 1):

quasi-homogeneous polynomial of order 3 →

$$\begin{pmatrix} a_{03} y^3 + a_{11} x y \\ b_{02} y^2 + b_{10} x \end{pmatrix}$$

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Example for type (2,1) quasi-homogeneity (the exponent of x is multiplied by 2, the one of y by 1):

quasi-homogeneous polynomial of order 2 → $b_{02}y^2 + b_{10}x$

$$\begin{pmatrix} a_{03}y^3 + a_{11}xy \\ b_{02}y^2 + b_{10}x \end{pmatrix}$$

Quasi-homogeneus polynomial decomposition

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Example for type (2,1) quasi-homogeneity (the exponent of x is multiplied by 2, the one of y by 1):

quasi-homogeneous polynomial vector of degree 1 →

$$\begin{pmatrix} a_{03} y^3 + a_{11} x y \\ b_{02} y^2 + b_{10} x \end{pmatrix}$$

Quasi-homogeneus polynomial decomposition

(the vector field can be ordered by type (2,1) quasi-homogeneous vector polynomials of degree -1,0,1,2,...)

Proposition 2 System (3)-(4) can be written as a sum of quasi-homogeneous polynomial vectors fields $\mathbf{F}_r$ of type $(2,1)$ and degree r, as follows

$$\dot{\mathbf{x}} = \mathbf{F}^{\pm}_{-1}(\mathbf{x}) + \mathbf{F}^{\pm}_{0}(\mathbf{x}) + \cdots + \mathbf{F}^{\pm}_{r-1}(\mathbf{x}) + \mathbf{F}^{\pm}_{r}(\mathbf{x}) + \cdots$$

where the superscripts $\pm$ stand for $\pm x \geqslant 0$, and the most significant term is $\mathbf{F}^{\pm}_{-1}(\mathbf{x}) = (-y, \pm 1)^{\top}$.

Quasi-homogeneous polynomial decomposition

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It suffices to do the change of variables $(x, y) \to (\varepsilon^2 x, \varepsilon y)$ to get

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \varepsilon^{-1} \begin{pmatrix} -y \\ 1 \end{pmatrix} + \begin{pmatrix} a_{10}x + a_{02}y^2 \\ b_{01}y \end{pmatrix} + \varepsilon \begin{pmatrix} a_{03}y^3 + a_{11}xy \\ b_{02}y^2 + b_{10}x \end{pmatrix} + \varepsilon^2 \begin{pmatrix} a_{04}y^4 + a_{12}xy^2 + a_{20}x^2 \\ b_{03}y^3 + b_{11}xy \end{pmatrix} + \cdots$$

and afterthat, to put $\varepsilon = 1$.

Main results

Theorem 1 For any natural number $n \geqslant 1$, system (3)-(4) is, in a neighbourhood of the origin, topologically equivalent to a system of the form

$$\dot{\mathbf{x}} = \begin{pmatrix} -y + \sum_{k=1}^{n} \mu_{2k}^{\pm} y^{2k} \\ \pm 1 \end{pmatrix} + \sum_{k \geqslant 2n} \mathbf{G}_k^{\pm}(\mathbf{x}),$$

where the symbol '+' applies for $x \geqslant 0$ and the symbol '−' does for $x \leqslant 0$, and the terms $\mathbf{G}_k^{\pm}(\mathbf{x})$ are quasi-homogeneous polynomial vector fields of type $(2, 1)$ and degree k.

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normal form up to degree 2n-1
(separable, Hamiltonian structure)

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Main results

(some computational issues)

The coefficients $\mu_{2k}^{\pm}$ have involved expressions in terms of the original coefficients. These expressions reduce significantly by removing first the terms $a_{10}^{\pm}x$ through the linear change $y \to y - a_{10}^{\pm}x$, leading to the new coefficients (superscripts $\pm$ omitted for brevity)

$$\tilde{a}_{10} = 0, \quad \tilde{b}_{10} = a_{10}b_{01} + b_{10}, \quad \tilde{b}_{01} = a_{10} + b_{01},$$

while for $p + q \geqslant 2$, we have

$$\tilde{a}_{pq} = \sum_{r=0}^{p} \binom{p+q-r}{q} a_{r,p+q-r}\, a_{10}^{p-r}, \quad \tilde{b}_{pq} = \sum_{r=0}^{p} \binom{p+q-r}{q} \left(b_{r,p+q-r} - a_{10}\, a_{r,p+q-r}\right) a_{10}^{p-r}.$$

Main results

(some computational issues)

In terms of these new tilde-coefficients, where already $\tilde{a}_{10} = 0$, we get

$$\mu_2^+ = \tilde{a}_{02}^+ + \tilde{b}_{01}^+, \quad \mu_2^- = \tilde{a}_{02}^- - \tilde{b}_{01}^-,$$

and

$$\begin{aligned} 3\mu_4^+ &= 2\tilde{a}_{20}^+ + 3\tilde{a}_{04}^+ + \tilde{a}_{12}^+ + \tilde{a}_{02}^+ \left(5\tilde{a}_{03}^+ + \tilde{a}_{11}^+ - 5\tilde{a}_{02}^+\tilde{b}_{01}^+ + 2\tilde{b}_{02}^+ - 7(\tilde{b}_{01}^+)^2\right) \\ &\quad -\tilde{b}_{01}^+ \left(\tilde{b}_{02}^+ - 2\tilde{a}_{03}^+ + 2(\tilde{b}_{01}^+)^2 + \tilde{b}_{10}^+\right) + \tilde{b}_{11}^+ + 3\tilde{b}_{03}^+, \\ 3\mu_4^- &= 2\tilde{a}_{20}^- + 3\tilde{a}_{04}^- - \tilde{a}_{12}^- + \tilde{a}_{02}^- \left(5\tilde{a}_{03}^- - \tilde{a}_{11}^- + 5\tilde{a}_{02}^-\tilde{b}_{01}^- - 2\tilde{b}_{02}^- - 7(\tilde{b}_{01}^-)^2\right) \\ &\quad +\tilde{b}_{01}^- \left(-\tilde{b}_{02}^- - 2\tilde{a}_{03}^- + 2(\tilde{b}_{01}^-)^2 + \tilde{b}_{10}^-\right) + \tilde{b}_{11}^- - 3\tilde{b}_{03}^-, \end{aligned}$$

where we emphasize the possible simplification when some $\tilde{a}_{02}$ or $\tilde{b}_{01}$ vanishes.

Quasi-homogeneous normal form approach

A short review for the smooth case

Key Lemma If the vector fields $\mathbf{F}_l$ and $\mathbf{G}_m$ are quasi-homogeneous of type $\mathbf{p}$ and degree l and m respectively, then the Lie bracket

$$[\mathbf{F}_l, \mathbf{G}_m] = \mathrm{D}\,\mathbf{F}_l(\mathbf{x})\mathbf{G}_m(\mathbf{x}) - \mathrm{D}\,\mathbf{G}_m(\mathbf{x})\mathbf{F}_l(\mathbf{x})$$

is a quasi-homogeneous vector field of type $\mathbf{p}$ and degree $l+m$.

Quasi-homogeneous normal form approach

A short review for the smooth case

Assume that the smooth system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is already written as a sum of quasi-homogeneous terms $\mathbf{F}_i$ of degree i and type $\mathbf{p}$,

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_l(\mathbf{x}) + \mathbf{F}_{l+1}(\mathbf{x}) + \cdots + \mathbf{F}_{l+k}(\mathbf{x}) + \cdots = \sum_{i \geqslant l} \mathbf{F}_i(\mathbf{x}).$$

Next, the near-identity change of variables of the form

$$\mathbf{x} = \mathbf{w} + \mathbf{P}_r(\mathbf{w}),$$

where $\mathbf{w} = (u, v)$ and $\mathbf{P}_r(\mathbf{w})$ is a polynomial vector field of type $\mathbf{p}$ and degree r to be determined later, produces

$$\dot{\mathbf{w}} = [I + \mathrm{D}\,\mathbf{P}_r(\mathbf{w})]^{-1} \sum_{i \geqslant l} \mathbf{F}_i(\mathbf{w} + \mathbf{P}_r(\mathbf{w})).$$

Quasi-homogeneous normal form approach

A short review for the smooth case

By expanding the RHS around the origin, we obtain

$$\dot{\mathbf{w}} = \left[I - \mathrm{D}\,\mathbf{P}_r(\mathbf{w}) + \mathrm{D}\,\mathbf{P}_r(\mathbf{w})^2 - \cdots \right] \sum_{i \geqslant l} \left[\mathbf{F}_i(\mathbf{w}) + \mathrm{D}\,\mathbf{F}_i(\mathbf{w})\mathbf{P}_r(\mathbf{w}) + \cdots \right],$$

or equivalently,

$$\dot{\mathbf{w}} = \sum_{k=0}^{r-1} \mathbf{F}_{l+k}(\mathbf{w}) + \left[\mathbf{F}_{l+r}(\mathbf{w}) - \mathrm{D}\,\mathbf{P}_r(\mathbf{w})\mathbf{F}_l(\mathbf{w}) + \mathrm{D}\,\mathbf{F}_l(\mathbf{w})\mathbf{P}_r(\mathbf{w}) \right] + \mathcal{R}(\mathbf{w}),$$

where the remainder

$$\mathcal{R}(\mathbf{w}) = \sum_{k \geqslant r+1} \mathbf{G}_{l+k}(\mathbf{w})$$

contains all the terms of degree greater or equal to $l + r + 1$.

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or equivalently, quasi-homogeneous vector polynomials up to degree *l+r-1 (not changed)*

$$\dot{\mathbf{w}} = \sum_{k=0}^{r-1} \mathbf{F}_{l+k}(\mathbf{w}) + \left[\mathbf{F}_{l+r}(\mathbf{w}) - \mathrm{D}\,\mathbf{P}_r(\mathbf{w})\mathbf{F}_l(\mathbf{w}) + \mathrm{D}\,\mathbf{F}_l(\mathbf{w})\mathbf{P}_r(\mathbf{w})\right] + \mathcal{R}(\mathbf{w}),$$

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or equivalently,

new quasi-homogeneous vector polynomial of degree *l+r*

$$\dot{\mathbf{w}} = \sum_{k=0}^{r-1} \mathbf{F}_{l+k}(\mathbf{w}) + \boxed{\left[\mathbf{F}_{l+r}(\mathbf{w}) - \mathrm{D}\,\mathbf{P}_r(\mathbf{w})\mathbf{F}_l(\mathbf{w}) + \mathrm{D}\,\mathbf{F}_l(\mathbf{w})\mathbf{P}_r(\mathbf{w})\right]} + \mathcal{R}(\mathbf{w}),$$

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Quasi-homogeneous normal form approach

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or equivalently,

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where the remainder

$$\mathcal{R}(\mathbf{w}) = \sum_{k \geqslant r+1} \mathbf{G}_{l+k}(\mathbf{w})$$

$$\mathbf{F}_{l+r}(\mathbf{w}) - [\mathbf{P}_r, \mathbf{F}_l](\mathbf{w})$$

contains all the terms of degree greater or equal to $l + r + 1$.

Quasi-homogeneous normal form approach

A short review for the smooth case

Thus, in order to remove as much terms as possible of degree $l + r$ we must consider the partial or total compatibility of the *homological equation*

$$\mathcal{L}[\mathbf{P}_r(\mathbf{w})] := [\mathbf{P}_r, \mathbf{F}_l](\mathbf{w}) = \mathbf{F}_{l+r}(\mathbf{w}),$$

where we assume fixed the term $\mathbf{F}_l$.

So far, the standard normal formal approach for smooth systems!

Sketch of the proof of Theorem 1

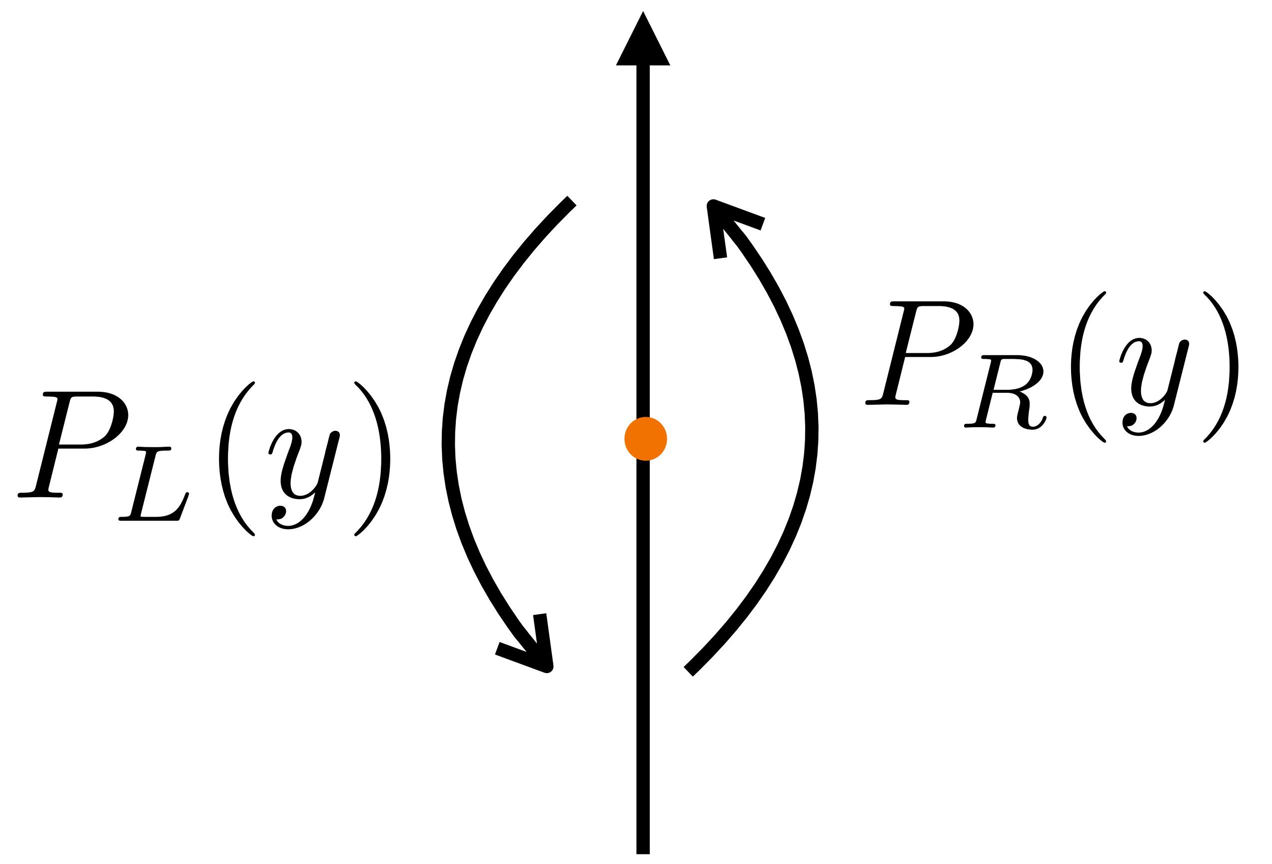
(Adapting quasi-homogeneous normal forms for pseudo-focus analysis)

- We have to work for each side separately.
- As we need to preserve the points at the discontinuity manifold, only some possible transformations are admissible.
- Therefore, for sake of simplicity, we use near-identity transformations not changing the second variable at all.

From the normal form to the return maps

Some results on return maps around a pseudo-focus

(Extended return maps are proper analytical involutions at the origin)



We can extend the half-return maps by considering their inverses and writing

$$\tilde{P}_L(y) = \begin{cases} P_L(y), & \text{if } y > 0, \\ 0, & \text{if } y = 0, \\ P_L^{-1}(y), & \text{if } y < 0, \end{cases} \qquad \tilde{P}_R(y) = \begin{cases} P_R^{-1}(y), & \text{if } y > 0, \\ 0, & \text{if } y = 0, \\ P_R(y) & \text{if } y < 0, \end{cases}$$

so that these new bi-directional maps become proper involutions, that is $\tilde{P}_L = \tilde{P}_L^{-1}$, and $\tilde{P}_R = \tilde{P}_R^{-1}$.

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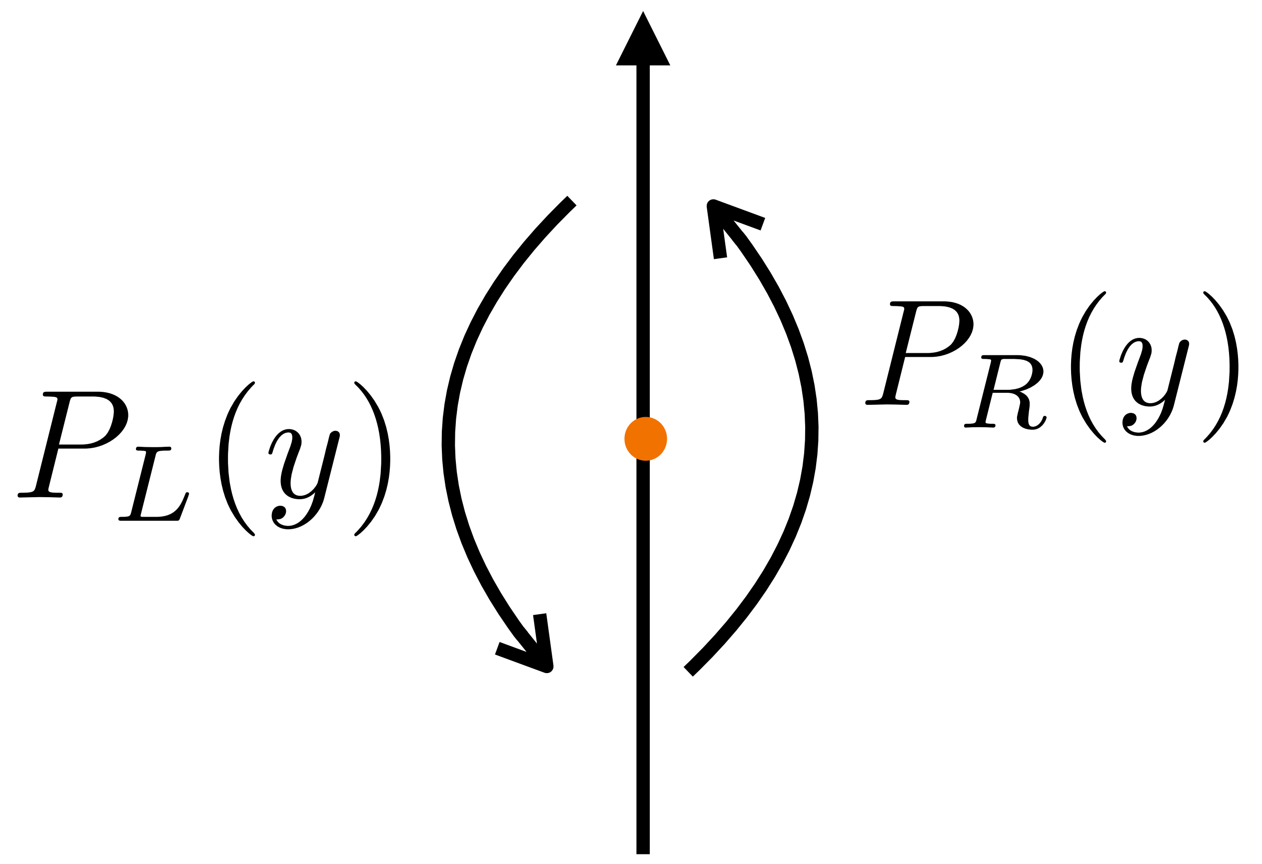

We can extend the half-return maps by considering their inverses and writing

$$\tilde{P}_L(y) = \begin{cases} P_L(y), & \text{if } y > 0, \\ 0, & \text{if } y = 0, \\ P_L^{-1}(y), & \text{if } y < 0, \end{cases} \qquad \tilde{P}_R(y) = \begin{cases} P_R^{-1}(y), & \text{if } y > 0, \\ 0, & \text{if } y = 0, \\ P_R(y) & \text{if } y < 0, \end{cases}$$

so that these new bi-directional maps become proper involutions, that is $\tilde{P}_L = \tilde{P}_L^{-1}$, and $\tilde{P}_R = \tilde{P}_R^{-1}$.

Proposition 3 For system (1) satisfying hypotheses (H1)-(H3) the functions $\tilde{P}_L$ and $\tilde{P}_R$ are proper analytical involutions at the origin.

Computing return maps from the normal form

Proposition 6. If we introduce for $k \geqslant 1$ the notation $\boldsymbol{\mu}_k^{\pm} = (\mu_2^{\pm}, \mu_4^{\pm}, \cdots, \mu_{2k-2}^{\pm}, \mu_{2k}^{\pm})$, then the analytical involutions at the origin corresponding to the maps $\tilde{P}_R$ and $\tilde{P}_L$ are

$$\tilde{P}_R(y) = -y + \sum_{k\geqslant 1} (\alpha_{2k}^{+} y^{2k} + \alpha_{2k+1}^{+} y^{2k+1}), \quad \tilde{P}_L(y) = -y + \sum_{k\geqslant 1} (\alpha_{2k}^{-} y^{2k} + \alpha_{2k+1}^{-} y^{2k+1}), \text{where}$$

$$\alpha_2^{\pm} = \frac{2}{3}\mu_2^{\pm}, \quad \alpha_3^{\pm} = \tilde{h}_3(\boldsymbol{\mu}_1^{\pm}) = -\frac{4}{9}(\mu_2^{\pm})^2, \quad \alpha_4^{\pm} = \frac{2}{5}\mu_4^{\pm} + h_4(\boldsymbol{\mu}_1^{\pm}) = \frac{2}{5}\mu_4^{\pm} + \frac{16}{27}(\mu_2^{\pm})^3.$$

$$\alpha_5^{\pm} = h_5(\boldsymbol{\mu}_2^{\pm}) = -\frac{64}{81}(\mu_2^{\pm})^4 - \frac{4}{5}\mu_2^{\pm}\mu_4^{\pm}, \ldots$$

In general, for $2 \leqslant k \leqslant n$, we have $\alpha_{2k}^{\pm} = \dfrac{2\mu_{2k}^{\pm}}{2k+1} + \tilde{h}_{2k}(\boldsymbol{\mu}_{k-1}^{\pm}), \quad \alpha_{2k+1}^{\pm} = \tilde{h}_{2k+1}(\boldsymbol{\mu}_k^{\pm})$,

being $\tilde{h}_{2k}$ and $\tilde{h}_{2k+1}$ polynomial functions such that $\tilde{h}_{2k}(\mathbf{0}) = \tilde{h}_{2k+1}(\mathbf{0}) = 0$.

Computing return maps from the normal form

Proposition 7 The difference map $\tilde{D}(y) = P_R^{-1}(y) - P_L(y)$ is given by

$$\tilde{D}(y) = \sum_{k\geqslant 1}(V_{2k}y^{2k} + V_{2k+1}y^{2k+1}),$$

where the coefficients $V_j = \alpha_j^+ - \alpha_j^-$ for $j \geqslant 2$, and the values $\alpha_j^\pm$ come from the above proposition. In particular,

$$V_2 = \frac{2}{3}(\mu_2^+ - \mu_2^-), \quad V_3 = -\frac{4}{9}\left((\mu_2^+)^2 - (\mu_2^-)^2\right),$$

and for $k \geqslant 2$

$$V_{2k} = \frac{2(\mu_{2k}^+ - \mu_{2k}^-)}{2k+1} + \tilde{h}_{2k}(\boldsymbol{\mu}_{k-1}^+) - \tilde{h}_{2k}(\boldsymbol{\mu}_{k-1}^-), \quad V_{2k+1} = \tilde{h}_{2k+1}(\boldsymbol{\mu}_k^+) - \tilde{h}_{2k+1}(\boldsymbol{\mu}_k^-),$$

where the functions $\tilde{h}_{2k}$ and $\tilde{h}_{2k+1}$ are the ones introduced before.

Computing return maps from the normal form

Remark. Given $p > 1$, when $\mu_{2k}^{+} = \mu_{2k}^{-}$ for $1 \leqslant k \leqslant p-1$, but $\mu_{2p}^{+} \neq \mu_{2p}^{-}$, that is $\boldsymbol{\mu}_{p-1}^{+} = \boldsymbol{\mu}_{p-1}^{-}$ and $\boldsymbol{\mu}_{p}^{+} \neq \boldsymbol{\mu}_{p}^{-}$, then $V_k = 0$ for $2 \leqslant k \leqslant 2p-1$ and so

$$\tilde{D}(y) = \sum_{k \geqslant p} (V_{2k} y^{2k} + V_{2k+1} y^{2k+1}), \text{ with } V_{2p} = \frac{2(\mu_{2p}^{+} - \mu_{2p}^{-})}{2p+1} \neq 0.$$

In this case, the origin is called a weak focus or order p, being stable (resp. unstable) if $V_{2p} < 0$ (resp. $V_{2p} > 0$). If for all $k \geqslant 1$ we have $\mu_{2k}^{+} = \mu_{2k}^{-}$, then the displacement map satisfies $\tilde{D}(y) \equiv 0$ in a neighborhood of the origin, which becomes a pseudo-center surrounded by a periodic annulus.

Application examples

1. A linear-quadratic system with pseudo-focus at the origin

$x < 0$	$x > 0$
$\dot{x} = t_- x - y$ $\dot{y} = -1 + d_- x$	$\dot{x} = t_+ x - y + \delta y^2$ $\dot{y} = 1 + d_+ x$

Application examples

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$\dot{x} = t_- x - y$ $\dot{y} = -1 + d_- x$	$\dot{x} = t_+ x - y + \delta y^2$ $\dot{y} = 1 + d_+ x$

$$
\begin{aligned}
\mu_2^+ &= \delta + t_+, \\
\mu_4^+ &= -\frac{t_+}{3}(d_+ + 2t_+^2 + 6t_+\delta + 5\delta^2), \\
\mu_6^+ &= \frac{t_+}{45}(9d_+^2 + 44d_+t_+^2 + 52t_+^4 + 94d_+t_+\delta + 260t_+^3\delta + 50d_+\delta^2 \\
&\quad +476t_+^2\delta^2 + 350t_+\delta^3 + 70\delta^4), \\
\mu_8^+ &= -\frac{t_+}{945}\big(135d_+^3 + 1176d_+^2t_+^2 + 3508d_+t_+^4 + 3392t_+^6 \\
&\quad +2184d_+^2t_+\delta + 14584d_+t_+^3\delta + 23744t_+^5\delta + 1017d_+^2\delta^2 \\
&\quad +22008d_+t_+^2\delta^2 + 67724t_+^4\delta^2 + 13600d_+t_+\delta^3 + 98224t_+^3\delta^3 \\
&\quad +2695d_+\delta^4 + 73766t_+^2\delta^4 + 25760t_+\delta^5 + 3080\delta^6\big).
\end{aligned}
$$

Application examples

1. A linear-quadratic system with pseudo-focus at the origin

$x < 0$	$x > 0$
$\dot{x} = t_- x - y$ $\dot{y} = -1 + d_- x$	$\dot{x} = t_+ x - y + \delta y^2$ $\dot{y} = 1 + d_+ x$

$$
\begin{aligned}
\mu_2^- &= -t_-, \\
\mu_4^- &= \frac{t_-}{3}\left(d_- + 2t_-^2\right), \\
\mu_6^- &= -\frac{t_-}{45}\left(d_- + 2t_-^2\right)\left(9d_- + 26t_-^2\right), \\
\mu_8^- &= \frac{t_-}{945}\left(d_- + 2t_-^2\right)\left(135d_-^2 + 906d_- t_-^2 + 1696t_-^4\right).
\end{aligned}
$$

$$
\begin{aligned}
\mu_2^+ &= \delta + t_+, \\
\mu_4^+ &= -\frac{t_+}{3}(d_+ + 2t_+^2 + 6t_+\delta + 5\delta^2), \\
\mu_6^+ &= \frac{t_+}{45}(9d_+^2 + 44d_+t_+^2 + 52t_+^4 + 94d_+t_+\delta + 260t_+^3\delta + 50d_+\delta^2 \\
&\quad +476t_+^2\delta^2 + 350t_+\delta^3 + 70\delta^4), \\
\mu_8^+ &= -\frac{t_+}{945}\left(135d_+^3 + 1176d_+^2t_+^2 + 3508d_+t_+^4 + 3392t_+^6\right. \\
&\quad +2184d_+^2t_+\delta + 14584d_+t_+^3\delta + 23744t_+^5\delta + 1017d_+^2\delta^2 \\
&\quad +22008d_+t_+^2\delta^2 + 67724t_+^4\delta^2 + 13600d_+t_+\delta^3 + 98224t_+^3\delta^3 \\
&\quad \left.+2695d_+\delta^4 + 73766t_+^2\delta^4 + 25760t_+\delta^5 + 3080\delta^6\right).
\end{aligned}
$$

Application examples

1. A linear-quadratic system with pseudo-focus at the origin

Proposition. For the discontinuous piecewise linear system that becomes when $\delta = 0$, the following statements are true.

(a) If $t_+ = t_- = 0$ then the system is piecewise Hamiltonian, and the origin is a pseudo-center.

(b) For $t_+ = -t_- \neq 0$ and $d_+ = d_-$ the origin is a pseudo-center whose periodic annulus is symmetric with respect to the y-axis.

(c) For $t_+ = -t_- \neq 0$ and $d_+ \neq d_-$ the origin is a weak pseudo-focus or order 2. In such a situation, there exists a small perturbation of any parameter (t_+, t_-) giving rise to one periodic orbit.

(d) If $t_+ + t_- \neq 0$ then the origin is a standard pseudo-focus.

Application examples

Proposition. When $\delta \neq 0$ the following statements hold.

(a) If $\delta + t_+ + t_- \neq 0$, then the origin is a standard pseudo-focus, being stable (resp. unstable) when $\delta + t_+ + t_- < 0$ (resp. > 0).

(b) When $\delta + t_+ + t_- = 0$, the origin is a pseudo-center in any of the two situations

(i) $(t_+, t_-, d_-) = (0, -\delta, -2\delta^2)$, (ii) $(t_+, t_-, d_+, d_-) = (-2\delta, \delta, 0, 0)$.

(c) If $\delta + t_+ + t_- = 0$, and $\rho = d_- t_- + d_+ t_+ + 2t_-^3 + t_+^3 + 4t_+^2 t_- + 5t_+ t_-^2 \neq 0$, then the origin is a weak focus of order 2, being stable (resp. unstable) when $\rho > 0$ (resp. $\rho < 0$). Then, there exist small perturbations of any parameter (δ, t_+, t_-) leading to the bifurcation of one limit cycle.

(d) If we take $(t_+, t_-, d_+, d_-) = \big(-(1 + \mu)\delta, \mu\delta, (\mu\lambda - 1)\delta^2, (\mu\lambda + \lambda - 2)\delta^2\big)$ for some new parameters (λ, μ), excluding the cases $\mu = -1$ and $\lambda = \mu = 1$ corresponding to statement (b), then the origin is a weak pseudo-focus of order 3 or even higher.

Concluding remarks

- A normal form has been proposed to analyze pseudo-equilibrium points of pseudo-focus type.
- This canonical form allows us to compute half Poincaré maps and characterize the orden of the pseudo-focus.
- Some linear-quadratic systems have been studied, obtaining families with a rich variety of behaviors.
- M. Esteban, E. Freire, E. Ponce and F. Torres, *On normal forms and return maps for pseudo-focus points*, J. Math. Anal. Appl. 507 (2022) 125774 (available online 25 October 2021

Gracias por su atención!