Principal Bautin ideal of monodromic singularities with inverse integrating factors

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Introduction

We consider families of real analytic planar differential systems

$$\dot{x} = P(x, y; \lambda), \quad \dot{y} = Q(x, y; \lambda), \tag{1}$$

or equivalently planar vector fields

$$\mathcal{X} = P(x, y; \lambda)\partial_x + Q(x, y; \lambda)\partial_y.$$

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- IN PRINCIPLE: Since $\mathcal{X}$ is analytic, independently I'lyashenko and Écalle, prove that the singularity only can be either a *center* or a *focus*.

Poincaré-Lyapunov center-focus problem

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To discern the subsets of Λ corresponding to a center and a focus.

The weighted polar blow-up

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Given $(p, q) \in W(\mathbf{N}(\mathcal{X}))$, we take the blow-up $(x, y) \mapsto (\rho, \varphi)$ given by

$$x = \rho^p \cos\varphi, \quad y = \rho^q \sin\varphi. \tag{2}$$

The differential equation on the cyclinder C

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$$\boxed{\frac{d\rho}{d\varphi} = \mathcal{F}(\varphi, \rho)}, \tag{3}$$

on the cylinder

$$C = \left\{(\varphi, \rho) \in \mathbb{S}^1 \times \mathbb{R} \,:\, 0 \leq \rho \ll 1\right\}, \text{ with } \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z}),$$

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- We define the set of (p, q)-characteristic directions $\Omega_{pq} = \{\varphi^* \in \mathbb{S}^1 : \Theta(\varphi^*, 0) = 0\}$.

The Poincaré map Π

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Let $\Phi(\varphi; \rho_0)$ be the solution of the Cauchy problem (3) with initial condition $\Phi(0; \rho_0) = \rho_0 > 0$ sufficiently small. Then we reparameterize Σ by $x = \rho_0^p$ so that

$$\Pi(\rho_0) = \Phi(2\pi, \rho_0).$$

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Medvedeva proves that Π has a Dulac asymptotic expansion:

$$\Pi(x) = \eta_1 x + \sum_j P_j(\log x) x^{\nu_j},$$

where $\eta_1 > 0$, the exponents $\nu_j > 1$ grow to infinity and the coefficients of the P_j are polynomials whose coefficients depend analytically on the coefficients of $\mathcal{X}$.

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Characterization of centers: $\Pi(x) = x$.

Inverse integrating factors on $C \backslash \Theta^{-1}(0)$

A not locally null real-valued $C^1(C \backslash \Theta^{-1}(0))$ function $V(\varphi, \rho)$ is an *inverse integrating factor* of (3) in $C \backslash \Theta^{-1}(0)$ if it is a solution of the linear partial differential equation

$$\boxed{\frac{\partial V}{\partial \varphi}(\varphi, \rho) + \frac{\partial V}{\partial \rho}(\varphi, \rho)\, \mathcal{F}(\varphi, \rho) = \frac{\partial \mathcal{F}}{\partial \rho}(\varphi, \rho)\, V(\varphi, \rho)}. \qquad (5)$$

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Laurent inverse integrating factors

Are inverse integrating factors that can be expanded in convergent Laurent series about $\rho = 0$:

$$V(\varphi, \rho) = \sum_{i \geq m} v_i(\varphi) \rho^i, \quad m \in \mathbb{Z}, \quad v_m(\varphi) \not\equiv 0, \tag{6}$$

whose coefficients are C^1 functions $v_i : \mathbb{S}^1 \backslash \Omega_{pq} \to \mathbb{R}$.

(p,q)-*critical parameters*

Given the weights $(p,q) \in W(\mathbf{N}(\mathcal{X}))$, we define the subset $\Lambda_{pq} \subset \Lambda$ of (p,q)-*critical parameters* as

$$\Lambda_{pq} = \{\lambda \in \Lambda : \Theta^{-1}(0) \backslash \{\rho = 0\} \neq \emptyset\}. \tag{7}$$

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Remark

- $\Omega_{pq} = \emptyset \Longrightarrow \Lambda_{pq} = \emptyset$.
- $\Lambda_{pq} = \emptyset \not\Longrightarrow \Omega_{pq} = \emptyset$.

The fundamental equation

Theorem 1

In the restricted parameter space $\Lambda \backslash \Lambda_{pq}$ equation

$$\boxed{V(0, \Pi(\rho_0)) = V(0, \rho_0)\, \Pi'(\rho_0)} \tag{8}$$

holds.

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Remark

- This Theorem was proved in the particular case that $\{\rho = 0\}$ is a periodic orbit, that is when $\Omega_{pq} = \emptyset$ by García, Giacomini and Grau in 2010.
- We have improved it allowing that $\{\rho = 0\}$ be a polycycle, that is, $\Omega_{pq} \neq \emptyset$ but we keeping in $\Lambda \backslash \Lambda_{pq}$.

Using Bruno's power geometry

Remark

We are going to use in (8) some results of the (two-dimensional) power geometry developed by Bruno designed to classify asymptotic expansions (including Puiseux series as particular case) of invariant branches at singularities of analytic planar vector fields.

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Remark

In the proof of the following theorem we re-prove, under the conditions of the theorem, the well-known fact that $\Pi(\rho) = \eta_1 \rho + o(\rho)$.

The structure of the Poincaré map

Theorem 2

Let the origin be a monodromic singularity of $\mathcal{X}$ with parameters in Λ and $0 \notin \Omega_{pq}$. Assume that equation (3) has a Laurent inverse integrating factor $V(\varphi, \rho)$ with multiplicity m. Then restricted to the parameter space $\Lambda \backslash \Lambda_{pq}$ the map Π has a formal power series expansion $\boxed{\Pi(\rho) = \sum_{i \geq 1} \eta_i \rho^i}$.

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- (i) If $m \leq 0$ then the origin is a center.
- (ii) If $m > 1$ then $\Pi(\rho) = \rho + \eta_m \rho^m (1 + O(\rho))$. In particular, if $\eta_m = 0$ then the origin is a center.

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(iii) If $m = 1$ and $\eta_1 = 1$ then the origin is a center.

The function $G(r)$

Taking the cylinder $C = I \times \mathbb{S}^1$ with $I = \{0\} \cup I^+$ and I^+ is a sufficiently small positive half-neighborhood of the origin, we define the function $G : I^+ \to \mathbb{R}$ by

$$\boxed{G(r) = \int_0^{2\pi} \frac{\mathcal{F}(\varphi, r)}{V(\varphi, r)} d\varphi.} \tag{9}$$

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(ii) In particular, G is well defined in $\Lambda \backslash \Lambda_{pq}$.

How to compute η_m

Theorem 3

Under the conditions of Theorem 2, $G(r) = \mathfrak{g} \in \mathbb{R}$ is a constant in I^+ and $\log \eta_1 = \mathfrak{g}$ when $m = 1$ and $\eta_m = \mathfrak{g}$ if $m > 1$.

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Remark

- Although $\mathfrak{g} = \lim_{r \to 0^+} G(r)$ we remark that when $\Omega_{pq} \neq \emptyset$ then, in general,

$$\lim_{r \to 0^+} G(r) \neq \int_0^{2\pi} \lim_{r \to 0^+} \frac{\mathcal{F}(\varphi, r)}{V(\varphi, r)} d\varphi.$$

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$$\lim_{r \to 0^+} G(r) \neq \int_0^{2\pi} \lim_{r \to 0^+} \frac{\mathcal{F}(\varphi, r)}{V(\varphi, r)} d\varphi.$$

- In general, G cannot be continuously extended to the origin. Indeed $G(0)$ does not exist when $m > 1$ whereas for $m = 1$ one gets $G(0) = PV \int_0^{2\pi} \mathcal{F}_1(\varphi) d\varphi$, which may exists or not and even when exists it can be different from $\mathfrak{g}$.

The Bautin ideal

- Let $E \subset \Lambda$ be an open subset such that $\mathbf{N}(\mathcal{X}|_E)$ is fixed.
- The Poincaré-Lyapunov quantities $\eta_i(\lambda)$ are analytic functions on E.
- Given the formal power series $\Pi(\rho; \lambda) = \sum_{i \geq 1} \eta_i(\lambda)\rho^i$, the Bautin ideal defined as $\mathcal{B} = \langle \eta_1 - 1, \eta_2, \eta_3, \ldots, \rangle$ is finitely generated.

The analiticity of Π and the cyclicity of $\mathcal{X}|_E$

Theorem 4

Under the assumptions of Theorem 2, the Bautin ideal $\mathcal{B}$ of $\mathcal{X}|_E$ is principal and given by $\mathcal{B} = \langle \eta_1(\lambda) - 1 \rangle$ if $m = 1$ or $\mathcal{B} = \langle \eta_m(\lambda) \rangle$ when $m \geq 2$.

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Moreover, the following holds:

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(ii) If $m = 1$ then $\Pi(\rho)$ is analytic.

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(iii) If $m > 1$ and the residue $\mathrm{Res}(1/V(0, \rho), 0) = 0$ then $\Pi(\rho)$ is analytic at $\rho = 0$.

Positive cyclicity when $E \neq \Lambda$

- We consider the 1-parameter family

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- Inverse integrating factor $v(x,y;\lambda) = (x^2+y^2)f(x,y;\lambda)$ where $f(x,y;\lambda) = x^2+y^2-\lambda$.

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- $\Lambda = \mathbb{R}$: When $\lambda \neq 0$, the system has a non-degenerate focus at the origin whose stability depends on the sign of λ. When $\lambda = 0$ it has a unstable degenerate (orbitally linearizable) focus at the origin.

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- For $\lambda > 0$, $f(x, y; \lambda) = 0$ is limit cycle which bifurcates from the origin. Contradicts Theorem 4?

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- For $\lambda > 0$, $f(x, y; \lambda) = 0$ is limit cycle which bifurcates from the origin. Contradicts Theorem 4?
- $E = \mathbb{R}^+$: Although $W(\mathbf{N}(\mathcal{X})) = \{(1, 1)\}$, one has that $\mathbf{N}(\mathcal{X})$ is not invariant in Λ because it is formed by the edge with endpoints $(2, 0)$ and $(0, 2)$ when $\lambda > 0$ whereas the edge changes having now endpoints $(4, 0)$ and $(0, 4)$ when $\lambda = 0$.

Discontinuity of η_m at $\lambda \in \partial E$

Using polar coordinates:

$$\dot{\rho} = \rho(-\lambda + \rho^2), \quad \dot{\varphi} = \lambda + \rho^2$$

and, using Bautin method:

$$\eta_1(\lambda) = \begin{cases} \exp(-2\pi) & \text{if} \quad \lambda > 0, \\ \exp(2\pi) & \text{if} \quad \lambda = 0. \end{cases}$$

MANY THANKS

FOR YOUR ATTENTION !!