The Poincaré map at singularities of planar vector fields

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Introduction

We consider families of real analytic planar differential systems

$$\dot{x} = P(x, y; \lambda), \quad \dot{y} = Q(x, y; \lambda), \tag{1}$$

or equivalently planar vector fields

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- $(x, y) = (0, 0)$ is a monodromic singularity of $\mathcal{X}$, that is local orbits turn around the origin for any $\lambda \in \Lambda \subset \mathbb{R}^p$.

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- $(x, y) = (0, 0)$ is a monodromic singularity of $\mathcal{X}$, that is local orbits turn around the origin for any $\lambda \in \Lambda \subset \mathbb{R}^p$.
- The characterization of the monodromic set Λ is usually done by the blow-up process (or σ-process) developed by Dumortier.

Stability and center-focus problem

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Poincaré-Lyapunov center-focus problem

To discern the subsets of Λ corresponding to a center and a focus.

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- DEGENERATE CASE: When $D\mathcal{X}(0,0) \equiv 0$ the stability and center-focus problem remains open except few specific cases.

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- The $\mathrm{Mo}^{(p,q)}$ monodromic class is the trivial one because the classical Poincaré– Lyapunov approach to compute the first terms in the Taylor expansion of Π works with minor modifications.
- We focus in the case when Γ contains singularities of the vector field.

The Poincaré map Π

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In this work we obtain one explicit formula for η that unifies the various expressions found in the literature of the different monodromic classes studied up to now.

The Newton diagram of $\mathcal{X}$

Given an analytic vector field $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$ with

$$P(x,y) = \sum_{(i,j)\in\mathbb{N}^2} \alpha_{ij} x^i y^{j-1}, \quad Q(x,y) = \sum_{(i,j)\in\mathbb{N}^2} \beta_{ij} x^{i-1} y^j,$$

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$W(\mathbf{N}(\mathcal{X})) \subset \mathbb{N}^2$ is the set containing all the weights associated to the edges in $\mathbf{N}(\mathcal{X})$.

The Newton diagram of $\mathcal{X}$

We consider a monodromic singularity at the origin of $\mathcal{X}$ with fixed Newton diagram $\mathbf{N}(\mathcal{X})$ with weights $W(\mathbf{N}(\mathcal{X})) = \{(p_1, q_1), (p_2, q_2), \ldots, (p_\ell, q_\ell)\}$ ordered by $q_1/p_1 < q_2/p_2 < \cdots < q_\ell/p_\ell$. In this way the edge i has weights (p_i, q_i) and the its upper connecting edge is the edge $i-1$.

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We say that the vertex of $\mathbf{N}(\mathcal{X})$ connecting the edges $i-1$ and i and having vector coefficient $(\mathfrak{a}_i, \mathfrak{b}_i)$ is a *nondegenerate vertex* if $p_i \mathfrak{b}_i - q_i \mathfrak{a}_i \neq 0$ and $p_{i-1} \mathfrak{b}_i - q_{i-1} \mathfrak{a}_i \neq 0$.

The weighted polar blow-up

For each $(p_i, q_i) \in W(\mathbf{N}(\mathcal{X}))$ we have a (p_i, q_i)-quasihomogeneous expansions of $\mathcal{X} = \mathcal{X}_{r_i} + \cdots$, where the leading vector field $\mathcal{X}_r = P_{p_i+r_i}(x, y)\partial_x + Q_{q_i+r_i}(x, y)\partial_y$ is a (p_i, q_i)-quasihomogeneous vector field of degree r_i.

Associated to the weights (p_i, q_i) we perform the *weighted polar blow-up* $(x, y) \mapsto (\rho, \varphi)$ given by

$$(x, y) = (\rho^{p_i} \cos\varphi, \rho^{q_i} \sin\varphi), \qquad (2)$$

transforming (1) into the differential equation

$$\frac{d\rho}{d\varphi} = \frac{R_i(\varphi, \rho)}{\Theta_i(\varphi, \rho)} = \frac{F_{r_i}(\varphi)\rho + O(\rho^2)}{G_{r_i}(\varphi) + O(\rho)}, \quad i = 1, \ldots, \ell. \qquad (3)$$

The weighted polar blow-up

The differential equation (3) is defined in $C \backslash \Theta_i^{-1}(0)$ being the cylinder $C = \left\{(\varphi, \rho) \in \mathbb{S}^1 \times \mathbb{R} \,:\, 0 \leq \rho \ll 1\right\}$ with $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$.

$\varphi = \varphi^*$ is a (p_i, q_i)-*characteristic direction* for the origin of $\mathcal{X}$ if $G_{r_i}(\varphi^*) = 0$ and we define $\Omega_{p_i q_i} \subset \mathbb{S}^1$ as the set of all the (p_i, q_i)-characteristic directions.

By monodromy, G_{r_i} is sign-defined on $\mathbb{S}^1$ and $\Omega_{p_i q_i} = G_{r_i}^{-1}(0)$.

The *Cauchy principal value* of an improper integral

Given a continuous function f defined in $I \subset [0, 2\pi] \backslash \Omega$ with $\Omega = \{\theta_1^*, \ldots, \theta_\ell^*\}$, the Cauchy principal value of the integral $\int_I f(\theta)\, d\theta$ is defined as

$$PV \int_I f(\theta)\, d\theta = \lim_{\varepsilon \to 0^+} \int_{I_\varepsilon} f(\theta)\, d\theta,$$

when the limit exists. Here we have used the notation $I_\varepsilon = I \backslash J_\varepsilon$ with $J_\varepsilon = \cup_{i=1}^{\ell} (\theta_i^* - \varepsilon, \theta_i^* + \varepsilon)$.

Related to equation (3), we define the quantities

$$\xi_{p_i q_i} = PV \int_0^{2\pi} \frac{F_{r_i}(\varphi)}{G_{r_i}(\varphi)} d\varphi, \quad i = 1, \ldots, \ell, \tag{4}$$

if they exist, and they will play a fundamental role in the results presented below.

The main result

The goal of this work is to give explicit conditions that guarantee that the linear part of the Poincaré map $\Pi(x) = \eta x + o(x)$ has leading coefficient

$$\eta = \exp\left(\pm \sum_{i=1}^{\ell} \lambda_i \, \xi_{p_i q_i} \right), \tag{5}$$

with

$$\lambda_1 = 1, \qquad \lambda_i = \frac{p_i \mathfrak{b}_i - q_i \mathfrak{a}_i}{p_{i-1} \mathfrak{b}_i - q_{i-1} \mathfrak{a}_i}, \quad i = 2, \ldots, \ell, \tag{6}$$

where $(\mathfrak{a}_i, \mathfrak{b}_i)$ is the vector coefficient of the vertex connecting the edges $i-1$ and i for $i = 2, \ldots, \ell$. In the formula (5) the positive sign is taken when the flow rotates counterclockwise (that is $G_{r_i}(\varphi) \geq 0$ in $\mathbb{S}^1$ for all $i = 1, \ldots, \ell$) and the negative sign otherwise.

Parameterizations of the transversal section

We reparameterize the transversal section $\Sigma = \{(x_0, 0) \in \mathbb{R}^2 : 0 < x_0 \ll 1\}$ where Π is defined by $x_0 = \rho_0^{p_i}$. Next we consider the solution $\Phi_i(\varphi; \rho_0)$ of the Cauchy problem (3) with initial condition $\Phi_i(0; \rho_0) = \rho_0 > 0$ sufficiently small.

We define their associated Poincaré map $\Pi_i(\rho_0) = \Phi_i(2\pi, \rho_0)$ so that $\Pi(x_0) = (\Pi_i(\rho_0))^{p_i}$. The next theorem relates the leading linear terms of Π_i and Π_j with $i \neq j$ under some nondegenerate conditions.

The difficulty of proof expression (5)

Theorem

Let $\mathcal{X}$ having a monodromic singular point at the origin and consider two arbitrary weights $\{(p_1, q_1), (p_2, q_2)\} \subset W(\mathbf{N}(\mathcal{X}))$. The Poincaré maps $\Pi_1(\rho_0)$ and $\Pi_2(r_0)$ written in weighted polar coordinates (ρ, φ) and (r, ψ) associated to the chosen weights are composition of regular transition maps and maps $\Delta_i^\varepsilon : \{\varphi_i^* - \varepsilon\} \to \{\varphi_i^* + \varepsilon\}$ and $\delta_j^\varepsilon : \{\psi_j^* - \varepsilon\} \to \{\psi_j^* + \varepsilon\}$ corresponding to the passage near the singularities on the polycycles $\{\rho = 0\}$ and $\{r = 0\}$, respectively, with characteristic directions $\varphi_i^* \in \Omega_{p_1 q_1}$ and $\psi_j^* \in \Omega_{p_2 q_2}$.

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Theorem

We assume that the following conditions hold:

(i) The Cauchy principal values $\xi_{p_i q_i}$ exist for $i = 1, 2$.

(ii) All the maps $\Delta_i^\varepsilon(\rho_0)$ and $\delta_j^\varepsilon(r_0)$ have linear leading terms whose limit when $\varepsilon \to 0^+$ exist for $i = 1, \ldots, \#\Omega_{p_1 q_1}$ and $j = 1, \ldots, \#\Omega_{p_2 q_2}$.

Then there are γ_1 and γ_2 such that the Poincaré maps $\Pi_1(\rho_0) = \eta_1 \rho_0 + o(\rho_0)$ and $\Pi_2(r_0) = \eta_2 r_0 + o(r_0)$ have leading coefficients

$$\eta_1 = \exp(\gamma_1 + \xi_{p_1 q_1}), \quad \eta_2 = \exp(\gamma_2 + \xi_{p_2 q_2}), \tag{7}$$

related by $\eta = \eta_1^{p_1} = \eta_2^{p_2}$.

The difficulty of proof expression (5)

Remark

A consequence of the previous Theorem is that $p_1(\gamma_1 + \xi_{p_1 q_1}) = p_2(\gamma_2 + \xi_{p_2 q_2})$ by condition $\eta_1^{p_1} = \eta_2^{p_2}$. But this relation does not imply the existence of λ_2 such that

$$\eta = \exp\left(\xi_{p_1 q_1} + \lambda_2\, \xi_{p_2 q_2}\right). \tag{8}$$

Anyway we want to emphasize that formula (8), which is a particular case of (5) restricted to $\#W(\mathbf{N}(\mathcal{X})) = 2$, is true in all the known monodromic classes as far as we know.

Conjecture

Clearly $\mathbf{N}(\mathcal{X})$ is not coordinate free, that is, in general $\mathbf{N}(\mathcal{X}) \neq \mathbf{N}(\phi_* \mathcal{X})$ for some analytic diffeomorphism ϕ around the origin of $\mathbb{R}^2$. Therefore we establish the following conjecture.

Conjecture

Given any analytic vector field $\mathcal{X}$ with a monodromic singular point, there are analytic coordinates such that the leading coefficient η of the asymptotic Dulac expansion $\Pi(x) = \eta x + o(x)$ of the Poincaré map Π has the form (5) provided $\xi_{p_i q_i}$ exists for all $(p_i, q_i) \in \mathbf{N}(\mathcal{X})$ and all the vertices of $\mathbf{N}(\mathcal{X})$ are nondegenerate.

All the known monodromic classes satisfy the conjecture

The conjecture is true in all the monodromic classes appearing in the literature.

Theorem

Let $\mathcal{X}$ be an analytic planar vector field having a monodromic singular point at the origin. In the following monodromic classes formula (5) holds: the $\mathrm{Mo}^{(p,q)}$ monodromic class; the $S_{3\omega}$ monodromic class; the $\mathcal{G}$-monodromic class with cubic first non-vanishing jet; the Mañosa monodromic class; the M_Γ monodromic class where $W(\mathbf{N}(\mathcal{X})) = \{(p_1, q_1), (p_2, q_2)\}$ with $q_1/p_1 < q_2/p_2$ and either q_1 and p_2 are even or q_2 is even, $p_1 = 1$, and q_1 is odd; the M_Γ monodromic class when $\#W(\mathbf{N}(\mathcal{X})) \geq 1$ and p_i or q_i is even for all $(p_i, q_i) \in W(\mathbf{N}(\mathcal{X}))$.

Example: Mañosa's monodromic family

Victor Mañosa shows that family

$$\dot{x} = xy^2 - y^3 + ax^5, \quad \dot{y} = 2x^7 - x^4y + 4xy^2 + y^3, \qquad (9)$$

has a monodromic singularity at the origin with parameters $\Lambda = \{a \in \mathbb{R} : \Delta(a) := 32 - (1 + 3a)^2 > 0\}$. Moreover he proves:

Mañosa's family in Λ

The origin is always a focus.

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Mañosa's proof:
i) Using the involved blow-up technique The Poincaré map is $\Pi(x) = \eta_1 x + o(x)$ with

$$\eta_1 = \exp\left(\pi + \frac{4\pi a}{\sqrt{\Delta(a)}}\right) \neq 1 \text{ if } a \neq -31/25. \qquad (10)$$

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ii) When $a = -31/25$ he uses a Lyapunov function.

The Mañosa monodromic class

The Mañosa monodromic class is characterized by the analytic vector fields $\mathcal{X} = \left(\sum_{i+j \geq k} p_{i,j} x^i y^j\right) \partial_x + \left(\sum_{i+j \geq k} q_{i,j} x^i y^j\right) \partial_y$ that satisfy:

(i) In polar coordinates $G_r(\varphi) \geq 0$ with $r = k - 1$ and $\Omega_{11} = \{0, \pi\}$;

(ii) $0 = q_{k,0} = q_{k-1,1} = p_{k,0} = q_{k+1,0} = q_{k+2,0} = q_{k,1} = p_{k+1,0} = q_{k+3,0}$;

(iii) $\Lambda(j) := q_{k-2,2} - j p_{k-1,1} > 0$ with $j = 1, 2, 3$, and $D := (q_{k+1,1} - 3p_{k+2,0})^2 - 4q_{k+4,0}(q_{k-2,2} - 3p_{k-1,1}) < 0$.

The Poincaré return map $\Pi(x) = \eta x + o(x)$ has the coefficient

$$\eta = \exp(\xi_{11} + \mathrm{M}_{11}), \tag{11}$$

where $\xi_{11} = PV \int_0^{2\pi} F_r(\varphi)/G_r(\varphi) d\varphi$, and

$$\mathrm{M}_{11} = 2\pi \frac{\Lambda(3)}{\Lambda(1)} \frac{2p_{k+2,0}\Lambda(3) - p_{k-1,1}(q_{k+1,1} - 3p_{k+2,0})}{\Lambda(3)\sqrt{-D}}. \tag{12}$$

The Mañosa monodromic class

The following proposition is proved for the whole Mañosa monodromic class.

Proposition

Let $\mathcal{X} = \left(\sum_{i+j\geq k} p_{i,j} x^i y^j\right) \partial_x + \left(\sum_{i+j\geq k} q_{i,j} x^i y^j\right) \partial_y$ be any analytic planar vector field having a singular point at the origin in the Mañosa monodromic class. Then $\mathbf{N}(\mathcal{X})$ has two edges with $W(\mathbf{N}(\mathcal{X})) = \{(1,1),(1,3)\}$ and its interior vertex has coefficient vector $(\mathfrak{a}, \mathfrak{b}) = (p_{k-1,1}, q_{k-2,2})$. Moreover, formula (8) works and is given by

$$\eta = \exp\left(\xi_{11} + \frac{\mathfrak{b} - 3\mathfrak{a}}{\mathfrak{b} - \mathfrak{a}} \xi_{13}\right), \tag{13}$$

assuming the flow rotates counterclockwise.

The Mañosa monodromic class

Doing the trigonometric change of variables $\varphi \mapsto m$ with $m = \sin\varphi/\cos^3\varphi$ associated to the weights $(1,3)$ we can express

$$\xi_{13} = PV\int_0^{2\pi} \frac{F_{k+1}(\varphi)}{G_{k+1}(\varphi)}\,d\varphi = PV\int_0^{2\pi} \frac{A(\varphi)}{\cos\varphi\, B(\varphi)}\,d\varphi$$

$$= 2\,PV\int_{-\infty}^{\infty} R(m)\,dm$$

where R is given by the rational function

$$R(m) = \frac{p_{k-1,1}\,m + p_{k+2,0}}{(q_{k-2,2} - 3p_{k-1,1})m^2 + (q_{k+1,1} - 3p_{k+2,0})m + q_{k+4,0}}.$$

The Mañosa monodromic class

The integral appearing in the last expression of ξ_{13} can be computed and gives

$$\xi_{13} = 2\pi \frac{2p_{k+2,0}(q_{k-2,2} - 3p_{k-1,1}) - p_{k-1,1}(q_{k+1,1} - 3p_{k+2,0})}{(q_{k-2,2} - 3p_{k-1,1})\sqrt{4q_{k+4,0}(q_{k-2,2} - 3p_{k-1,1}) - (q_{k+1,1} - 3p_{k+}}}$$

in agreement with equation (12) of work of Mañosa.

The Mañosa monodromic class

The integral appearing in the last expression of ξ_{13} can be computed and gives

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in agreement with equation (12) of work of Mañosa.
Recall that $\eta = \exp(\xi_{11} + \mathrm{M}_{11})$ by (11) where M_{11} is defined in (12). It is easy to see that

$$\mathrm{M}_{11} = \frac{\mathfrak{b} - 3\mathfrak{a}}{\mathfrak{b} - \mathfrak{a}} \xi_{13},$$

finishing the proof.

MANY THANKS

FOR YOUR ATTENTION !!