On the distribution of the zeros of some polynomial maps $(P, Q) \colon \mathbb{R}^2 \to \mathbb{R}^2$

JAUME LLIBRE
Universitat Autònoma de Barcelona

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This is a joint work with Claudia Valls

Chavarriga 1997

Chavarriga 1997

Chavarriga-Giné 1999

Chavarriga 1997

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Chavarriga 2005

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Giné-Garcia-Grau 2010

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Giné-Garcia-Grau-Maza 2012

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Chavarriga-Giné 1999

Chavarriga 2005

Giné-Garcia-Grau 2010

Giné-Garcia-Grau-Maza 2012

Up to now with Jaume Giné up we have published together 60 papers.

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We assume that the map (P, Q) has exactly nm different real zeros.

The objective of the paper is to study the distribution of the nm zeros of the map (P, Q) in the plane $\mathbb{R}^2$ when $n = 1, 2, 3$ and $m \leq 4$.

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Note that there exists a non-negative integer q such that $A_q \neq \emptyset$ and $A_{q+1} = \emptyset$.

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We also say that the zeros of (P, Q) belonging to A_i are on the i-th level.

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Moreover there exist examples of such maps whose zeros have these distributions.

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A. N. Berlinskii, On the behavior of the integral curves of a differential equation, Izv. Vyssh. Uchebn. Zaved. Mat. **2** (1960), 3–18.

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A. Cima, A. Gasull and F. Mañosas, Some applications of the Euler-Jacobi formula to differential equations, Proc. Amer. Math. **118** (1993), 151–163.

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$$\sum_{a \in A} \frac{R(a)}{J(a)} = 0,$$

where A is the set of the nm zeros of the polynomial map $(P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$. This is the Euler-Jacobi formula.

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4) Let $X = (P, Q)$ be a polynomial vector field of degree n. If X has $n + 1$ equilibrium points on a straight line then this line is full of equilibrium points.

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Moreover there exist examples of such maps whose zeros have these distributions.

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(c) If $|\sum_{a \in A} i(a)| = 2$ there are only the two configurations $(3; 1)$ with either $(3+, -)$ or $(3-, +)$.

There exist examples of quadratic polynomial differential systems with such configurations.

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Apply this formula being the polynomial R the straight line through the points P_1^+ and P_2^+:

$$\frac{R(P_1^-)}{J(P_1^-)} + \frac{R(P_2^-)}{J(P_2^-)} = 0.$$

$$\frac{R(P_1^-)}{J(P_1^-)} + \frac{R(P_2^-)}{J(P_2^-)} = 0.$$

Since $J(P_1^-) < 0$ and $J(P_2^-) < 0$ we have that $R(P_1^-)R(P_2^-) < 0$,

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Now apply the Euler-Jacobi Formula being the polynomial R the straight line through the points P_1^+ and P_1^-:

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$$\frac{R(P_2^+)}{J(P_2^+)} + \frac{R(P_2^-)}{J(P_2^-)} = 0.$$

Since $J(P_2^+) > 0$ and $J(P_2^-) < 0$ it follows that $R(P_2^+)R(P_2^-) > 0$,

$$\frac{R(P_2^+)}{J(P_2^+)} + \frac{R(P_2^-)}{J(P_2^-)} = 0.$$

Since $J(P_2^+) > 0$ and $J(P_2^-) < 0$ it follows that $R(P_2^+)R(P_2^-) > 0$, hence the points P_2^+ and P_2^- are in the same side of the straight line R.

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Similar arguments prove statement (c) of the Berlinskii's Theorem.

THANK YOU VERY MUCH FOR YOUR ATTENTION