Criticality via "first order" development of the period constants

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Main References

This talk is based in the joint works:

- I. Sánchez Sánchez, J. Torregrosa. Criticality via first order development of the period constants. Nonlinear Anal., 203, 112164:1-21. 2021.
- I. Sánchez Sánchez, J. Torregrosa. New lower bounds of the number of critical periods in reversible centers. J. Differential Equations, 292, 427-460. 2021.

The main technique is well described in

- J. Giné, L. F. d. S. Gouveia, J. Torregrosa. Lower bounds for the local cyclicity for families of centers. J. Differential Equations, 275, 309–331, 2021.

The (local) period function

Given an analytic center of type

$$\begin{cases} \dot{x} = -y + X(x, y, \lambda), \\ \dot{y} = x + Y(x, y, \lambda), \end{cases}$$

with X, Y polynomials starting at least with quadratic monomials. We can define the *Period Function* $T(\rho)$ as the least period of the trajectory passing trough $(\rho, 0)$.

The (local) period function

In polar coordinates we can write

$$\begin{cases} \dot{r} = \sum\limits_{k=2} \mathcal{R}_k(\theta) r^k, \\ \dot{\theta} = 1 + \sum\limits_{k=1} \Theta_k(\theta) r^k. \end{cases}$$

The return map can be defined from the solution of

$$\frac{dr}{d\theta} = \sum_{k=2} R_k(\theta) r^k, \qquad r(0) = \rho,$$

in series in ρ as

$$r(\theta, \rho) = \rho + u_2(\theta)\,\rho^2 + u_3(\theta)\,\rho^3 + \cdots .$$

Notice that, as we have a center at the origin, $u_k(2\pi) = 0$ for all k and

$$\Delta(\rho) = r(2\pi, \rho) - \rho \equiv 0.$$

The (local) period function

From the second equation, in polar coordinates,

$$\dot{\theta} = 1 + \sum_{k=1} \Theta_k(\theta) r^k,$$

and using the solution when we have a center ($u_k(2\pi) = 0$ for every k),

$$r_c(\theta, \rho) = \rho + u_2(\theta)\,\rho^2 + u_3(\theta)\,\rho^3 + \cdots .$$

Hence, we can compute the time map for each closed curve passing through $(\rho, 0)$ as

$$T(\rho) = \int_0^{2\pi} \frac{d\theta}{1 + \sum_{k=1} \Theta_k(\theta)(r_c(\theta,\rho))^k}.$$

The (local) period function

This function is analytic (at the origin) and it can be written as

$$T(\rho) = 2\pi\left(1 + \sum_{k=1}^{\infty} \widehat{T}_k(\lambda)\rho^k\right).$$

It is well-known that the first non vanishing $\widehat{T}_k(\lambda)$ has an even subscript.

Such coefficients, which are defined when the previous vanish, are known as the period constants $T_k(\lambda) = \widehat{T}_{2k}(\lambda)$.

When all T_k vanish we have an isochronous (or linearizable) center.

Criticality and related problems

Question

Which is the criticality (maximal number of oscillations of small amplitude of the period function) of the center equilibrium?

Question

Which classes of centers we are interested in?

Question

Depending on the family, which is the codimension of the problem?

Question

For a fixed polynomial family of degree n, which is the maximal value of ℓ such that $T_1 = \cdots = T_{\ell-1} = 0$, $T_\ell \neq 0$ but if $T_\ell = 0$ then we have a isochronous center. Can we **unfold** *in a fixed family?*

Lower bounds for the criticality of reversible centers

Theorem (SanTor2021)

The number of **local critical periods** *in the family of polynomial time-reversible centers of degree n is (at least)*

$$\mathcal{C}_\ell(n) \geq \begin{cases} 2^*, & \textit{for } n = 2 \textit{ (Chicone–Jacobs, 1989)}, \\ 6, & \textit{for } n = 3 \textit{ (Yu–Han, 2009)}, \\ 10, & \textit{for } n = 4, \\ 22, & \textit{for } n = 6, \\ (n^2+n-2)/2, & \textit{for } 5 \leq n \leq 9, \\ (n^2+n-4)/2, & \textit{for } 10 \leq n \leq 16. \end{cases}$$

Previous results (Lower bounds)

For fixed and small values of n

- Chicone and Jacobs, 1989, $n = 2$. (* $\mathcal{C}_\ell(2) = 2$
- Yu and Han, 2009, $n = 3$.

For any n

- Cima, Gasull, and Silva, 2008: $2[(n-2)/2]$.
- Gasull, Liu, and Yang, 2010: order $n^2/4$ ($(n^2+6n-4)/4$ even).

Lower bounds and number of free parameters

Is there any relation with the number of parameters?

Question

Is it true that $\mathcal{C}_\ell(n) = \dfrac{n^2+3n-6}{2}$ *for* $n \geq 3$ *?*

Not for Hamiltonian centers!

- Yu, Han, and Zhang, 2010: $\mathcal{C}_\ell(3) \geq 7$.
- Cen, 2021: $(n^2+2n-5)/2$ odd ($(n^2-4)/2$ even).
- De Maesschalck and Torregrosa, 2024: n^2-2 odd (n^2-2n-1 even).

First order developments and Implicit Function Theorem

Let $\lambda = (a_{20}, a_{11}, \ldots, b_{20}, b_{11}, \ldots) \in \mathbb{R}^N$. Consider a polynomial reversible perturbation of a reversible isochronous center (when $\lambda = 0$):

$$\begin{cases} \dot{x} = -y + X_c(x,y) + \sum_{k+l=2}^{n} a_{kl}\, x^k y^l, \\ \dot{y} = x + Y_c(x,y) + \sum_{k+l=2}^{n} b_{kl}\, x^k y^l. \end{cases}$$

Let us denote by $T_k^{[1]}(\lambda)$ the first-order truncation of the Taylor series, with respect to λ, of the period constants $T_k(\lambda)$.

Theorem

If the matrix of coefficients of $(T_1^{[1]}(\lambda), \ldots, T_{m+1}^{[1]}(\lambda))$ with respect to λ has rank $m+1$, the local criticality of the origin is at least m.

Question

Why we use isochronous centers? Which? Which are the best?

Melnikov method

Let $\lambda = (a_{20}, a_{11}, \ldots, b_{20}, b_{11}, \ldots) \in \mathbb{R}^N$. Consider the perturbation of an isochronous center (when $\lambda = 0$ and $\varepsilon = 0$)

$$\begin{cases} \dot{x} = -y + X_c(x,y) + \varepsilon \sum_{k+l=2}^{n} a_{kl}\, x^k y^l, \\ \dot{y} = x + Y_c(x,y) + \varepsilon \sum_{k+l=2}^{n} b_{kl}\, x^k y^l. \end{cases}$$

Let $T_k^{[1]}(\lambda)$ be the first-order truncation of the Taylor series, with respect to λ, of the period constants. Writing the Taylor series in ε of the period function of the above system as $T(\rho, \lambda, \varepsilon) = 2\pi + \sum_{k=1}^{\infty} \mathcal{T}_k(\rho, \lambda)\varepsilon^k$, then:

Theorem

For ρ small enough, the first averaging function $\mathcal{T}_1(\rho, \lambda)$ writes as

$$\mathcal{T}_1(\rho, \lambda) = \sum_{k=1}^{N} T_k^{[1]}(\lambda) \left(1 + \sum_{j=1}^{\infty} \alpha_{kj0} \rho^j \right) \rho^{2k}.$$

The parallelization technique

As $T_k^{[1]}(\lambda)$ are polynomials of degree 1 in λ, we can compute each coefficient separately.
For example the coefficients of $T_k^{[1]}(\lambda)$ corresponding to a_{20} can be obtained from the Taylor series of degree 1 of the system

$$\begin{cases} \dot{x} = -y + X_c(x,y) + a_{20}x^2, \\ \dot{y} = x + Y_c(x,y). \end{cases}$$

The linear part of the sum is the sum of the linear parts.

For most cases this technique is essential. Without it we can not get all necessary expressions of $T_k^{[1]}(\lambda)$ for proving the results.

The parallelization technique

Notice that the parallelization result does not depend on the particular mechanism used to compute $T_k(\lambda)$ or $T_k^{[1]}(\lambda)$.

We recommend to use a method that uses only linear algebra, as the one that uses the Lie bracket (for example). Because, the period constants can also be thought as the necessary conditions for a center be linearizable.

Explicit Cases

1. Reversible Quadratics (No free parameters). First and Higher order.
2. Linear center under Quadratic Reversible Perturbations.
3. "Higher degree Reversible families"?
   a. Holomorphic cubic (1 parameter). Explicit and simple.
   b. Holomorphic quartic (2 parameters). Where are the difficulties?

The reversible quadratic family (QR)

$$\begin{cases} \dot{x} = -y + a_{20}x^2 + a_{02}y^2, \\ \dot{y} = x + b_{11}xy. \end{cases}$$

$$T_1 = \frac{1}{24}(10a_{02}^2 + 10a_{02}a_{20} - a_{02}b_{11} + 4a_{20}^2 - 5a_{20}b_{11} + b_{11}^2),$$
$$T_2 = \frac{1}{48}(a_{20} + a_{02})a_{02}(30a_{02}^2 + 20a_{02}a_{20} + 15a_{02}b_{11} + 4a_{20}^2 + 2a_{20}b_{11}),$$
$$T_3 = \frac{1}{1800}a_{02}^2(4a_{02} + a_{20})(a_{20} + a_{02})(19565a_{02}^2 + 28190a_{02}a_{20} + 9867a_{20}^2).$$

The (nonlinear) isochronous systems

$$QR_1 = \{a_{02} = 0, a_{20} = 1, b_{11} = 4\}, QR_3 = \{a_{02} = -1, a_{20} = 1, b_{11} = 2\},$$

$$QR_2 = \{a_{02} = 0, a_{20} = 1, b_{11} = 1\}, QR_4 = \{a_{02} = -1/4, a_{20} = 1, b_{11} = 1/2\}.$$

The 1st-order perturbation of the isochronous QR (1)

$QR_1(\varepsilon) = \{a_{02} = \varepsilon_1, a_{20} = 1 + \varepsilon_2, b_{11} = 4 + \varepsilon_3\}$

$$
\begin{aligned}
T_1^{[1]} &= (2\varepsilon_1 - 4\varepsilon_2 + \varepsilon_3)/8, \\
T_2^{[1]} &= (4\varepsilon_2 - \varepsilon_3)/8, \\
T_3^{[1]} &= -(4\varepsilon_2 - \varepsilon_3)/8.
\end{aligned}
$$

Rank $= 2$, consequently 1 (local) critical period.

$QR_2(\varepsilon) = \{a_{02} = 0 + \varepsilon_1, a_{20} = 1 + \varepsilon_2, b_{11} = 1 + \varepsilon_3\}$

$$
\begin{aligned}
T_1^{[1]} &= (3\varepsilon_1 + \varepsilon_2 - \varepsilon_3)/8, \\
T_2^{[1]} &= (5\varepsilon_1 + \varepsilon_2 - \varepsilon_3)/16, \\
T_3^{[1]} &= (35\varepsilon_1 + 5\varepsilon_2 - 5\varepsilon_3)/128.
\end{aligned}
$$

Rank $= 2$, consequently 1 (local) critical period.

The 1st-order perturbation of the isochronous QR (2)

$QR_3(\varepsilon) = \{a_{02} = -1 + \varepsilon_1, a_{20} = 1 + \varepsilon_2, b_{11} = 2 + \varepsilon_3\}$:

$$\begin{aligned} T_1^{[1]} &= -(\varepsilon_1 + \varepsilon_2)/2, \\ T_2^{[1]} &= (\varepsilon_1 + \varepsilon_2)/2, \\ T_3^{[1]} &= -(\varepsilon_1 + \varepsilon_2)/2. \end{aligned}$$

Rank $= 1$, consequently 0 (local) critical period.

$QR_4(\varepsilon) = \{a_{02} = -1/4 + \varepsilon_1, a_{20} = 1 + \varepsilon_2, b_{11} = 1/2 + \varepsilon_3\}$:

$$\begin{aligned} T_1^{[1]} &= (18\varepsilon_1 + 12\varepsilon_2 - 15\varepsilon_3)/32, \\ T_2^{[1]} &= (70\varepsilon_1 + 64\varepsilon_2 - 93\varepsilon_3)/1024, \\ T_3^{[1]} &= (2310\varepsilon_1 + 2560\varepsilon_2 - 3965\varepsilon_3)/65536. \end{aligned}$$

Rank $= 2$, consequently 1 (local) critical period.

The higher-order perturbation of the isochronous QR

- QR_1, QR_2, QR_4

$$\begin{aligned} T_1 &= u_1 + O_2(u_1, u_2, u_3) = v_1, \\ T_2 &= u_2 + O_2(u_1, u_2, u_3) = v_2, \\ T_3 &= 0. \end{aligned}$$

  Only 1 critical period, with any order.

- QR_3

$$\begin{aligned} T_1 &= u_1 + O_2(u_1, u_2, u_3) = v_1, \\ T_2 &= u_2^2 + O_3(u_1, u_2, u_3) = v_2^2, \\ T_3 &= 0. \end{aligned}$$

  No critical periods up to first order and 1 with second order.

There are no more critical periods with higher order developments.

Solving directly $\{T_1 = T_2 = 0\}$ in QR

In all four families for Reversible Quadratics (QR_1, QR_2, QR_3, QR_4) we can study directly the solving substitution problem.

After solving $\{T_1 = 0, T_2 = 0\}$ with respect to $\{\varepsilon_1, \varepsilon_2\}$ we obtain 6 cases.

In 4 of them we get, by direct substitution, $T_3 \equiv 0$.

In the other two we get $T_3 \neq 0$, which says that we are far from the isochronous center (Contradiction!)

A more accurate analysis, and not direct computations, shows that, in fact, only one or two families are close to isochronous, and in all we get $T_3 = 0$.

There are no more critical periods with higher order developments.

There exists (at least) one curve of higher degeneracy:

$$L(\varepsilon) = \{a_{02} = -3\varepsilon, a_{20} = 5\varepsilon, b_{11} = 2\varepsilon\}.$$

$$T_1 = 0, \qquad T_2 = 0, \qquad T_3 = 63\varepsilon^6/10.$$

Consequently, we can get 2 (local) critical periods (but from the linear center), perturbing inside the reversible quadratics, if we can found a versal unfolding.

$$L(\varepsilon) = \{a_{02} = -3\varepsilon, a_{20} = (5+\lambda_2)\varepsilon, b_{11} = (2+\lambda_1)\varepsilon\}.$$

$$\begin{aligned}
T_1 &= (4\lambda_2^2 - 5\lambda_2\lambda_1 + \lambda_1^2 - 18\lambda_1)\varepsilon^2,\\
T_2 &= -(\lambda_2 + 2)(4\lambda_2^2 + 2\lambda_2\lambda_1 - 16\lambda_2 - 35\lambda_1)\varepsilon^4/46,\\
T_3 &= 3(\lambda_2 - 7)(\lambda_2 + 2)(3289\lambda_2^2 + 4700\lambda_2 - 30)\varepsilon^6/200.
\end{aligned}$$

Degree n reversible isochronous families

Holomorphic (isochronous) reversible centers of degree n:

$$\dot{z} = \mathrm{i}\, z \prod_{j=1}^{n-1} (1 - a_j z),$$

with $n > 1$ and $a_j \in \mathbb{R} \setminus \{0\}$

Question

Which is the criticality (perturbing inside the reversible class of degree n) of the origin?

We notice that when $n = 2$, we can assume $a_1 = -1$ ($\dot{z} = \mathrm{i}\, z + z^2$) and this case corresponds to QR_3.

Proofs for different n

Due to the computational difficulties we will see the proof for $n = 3$ to see the degeneracies and the main ideas that will appear for higher values of n. And because for $n = 3$ it can be seen better and explicitly why the technique applies.

For people that are familiar with degenerate cases for the analysis of the lower founds for the cyclicity in low degree planar vector fields here they will see a lot of similarities. In particular with the proof of the existence of 12 limit cycles in cubics.

But here we can get the computations of the generic and non generic cases much more explicit.

Cubic reversible family (main result)

It is not restrictive to assume $a_1 = 1$.

Proposition

Let $a \in \mathbb{R} \setminus \{0\}$. Consider the 1-parameter family of cubic (holomorphic) reversible systems

$$\dot{z} = \mathrm{i}\, z(1-z)(1-az).$$

The number of critical periods bifurcating from the origin when perturbing in the class of reversible cubic systems is at least 5 for $a \in \{-\frac{3}{2}, -1, -\frac{2}{3}, \frac{1}{2}, 2\}$ and 4 otherwise.

The relation with previous results

The described bifurcation phenomenon (the existence of more zeros than expected near generic situations) is close with previous used techniques:

- The existence of inflexion points in plots of centroid curves $(P(h), Q(h))$ in the analysis of Picard–Fuchs equations.
- The existence of ECT-families with accuracy.

Some of them are described in

C. Christopher, C. Li and J. Torregrosa. Limit cycles of differential equations. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser (2nd edition). February 2024.

Cubic reversible family (generic, rank 5)

When $a \in \mathbb{R} \setminus \{-3/2, -1, -2/3, 0, 1/2, 2\}$, the rank of the linear developments of first five period constants of this system with respect to $(r_{11}, r_{02}, r_{21}, r_{12}, r_{03})$ is 5. Consequently, at least 4 critical periods bifurcate from the origin because, using the Implicit Function Theorem, we have

$$T_k = u_k, \text{ for } k = 1, \ldots, 5.$$

Only with a first order analysis we have no more critical periods because 5 is the maximal rank.

Cubic reversible family (nongeneric but special, rank 4)

When $a \in \mathbb{R} \setminus \{-1, 0, 1/2, 2\}$, the rank of the linear developments of first five period constants of this system with respect to $(r_{11}, r_{02}, r_{21}, r_{12})$ is 4. Then, after using the Implicit Function Theorem, the period constants take the form

$$T_k = u_k, \text{ for } k = 1, \ldots, 4.$$

Taking $u_1 = u_2 = u_3 = u_4 = 0$ and $r_{03} = u_5$, we get

$$T_5 = \frac{5}{24}\,\frac{P(a)}{3a^2 + 2a + 3}\,u_5 + u_5^2 \sum_{j=0}^{\infty} f_j(a)\,u_5^j,$$

$$T_6 = -\frac{1}{42}\,\frac{Q(a)}{3a^2 + 2a + 3}\,u_5 + u_5^2 \sum_{j=0}^{\infty} g_j(a)\,u_5^j,$$

where $P(a) = a^3(a-2)(3a+2)(2a+3)(2a-1)$,
$Q(a) = a^3(a-2)(2a-1)(834a^2 + 1735a + 834)(a+1)^2$, and f_j and g_j are rational functions.

Cubic reversible family (nongeneric but special, rank 4)

When $P(a) \neq 0$ (the above generic case) we have 4 critical periods.

For the nongeneric special cases

$$a = -3/2, -2/3 \quad [P(a) = 0, P'(a) \neq 0, Q(a) \neq 0]$$

we have 5 (one more) critical periods.
Proved with the technique described in [GinGouTor2021].

Alternatively, we can also compute the development of T_k of order 2 and check that we have a new "free" parameter in T_5 being $T_6 \neq 0$.

It is important to restrict our parameter space in order that, after a necessary blowing up (in the parameter space), the higher order terms of T_5 do not affect in the computations and having a complete control that T_5 vanishes completely but not T_6.

Cubic reversible family (most degenerated, rank 3)

The remaining cases $a \in \{-1, 1/2, 2\}$ require a more accurate analysis.

Then the rank of the linear parts is only 3 and, similarly to what we did above, we have $T_k = u_k$, for $k = 1, 2, 3$ and we should study the second-order developments of T_4, T_5, T_6 under the condition $u_1 = u_2 = u_3 = 0$ with respect to the remaining parameters.

Quartic reversible family

Assuming that $a_1 = 1$ and writing $a_2 = a$ and $a_3 = b$ we have

Proposition

Let $a, b \in \mathbb{R}$. Consider the 2-parameter family of quartic (holomorphic) reversible systems

$$\dot{z} = \mathrm{i}\, z\,(1 - z)\,(1 - az)\,(1 - bz)\,.$$

Generically, at least 8 critical periods bifurcate from the origin when perturbing in the class of reversible quartic centers. Moreover, in this perturbation class there exists a point (a, b) such that at least 10 critical periods bifurcate from the origin.

Quartic reversible family (proof I)

After a linear change of coordinates in the parameters space we obtain that, generically, the period constants have the following form:

$$
\begin{aligned}
T_k &= u_k + O_2, \text{ for } k = 1, \ldots, 8, \\
T_9 &= \frac{G(a,b)P(a,b)}{D(a,b)} u_9 + O_2, \\
T_{10} &= \frac{G(a,b)Q(a,b)}{D(a,b)} u_9 + O_2, \\
T_{11} &= \frac{G(a,b)R(a,b)}{D(a,b)} u_9 + O_2,
\end{aligned}
$$

with $G(a,b) = (ab - a - b + 2)(ab - 2b^2 - a + b)(2a^2 - ab - a + b)a^3b^3$ and $P(a,b)$, $Q(a,b)$, $R(a,b)$, and $D(a,b)$ certain polynomials with rational coefficients in the variables a and b. The respectively total degrees are $37, 39, 41$, and 37. Their number of monomials are respectively $657, 736, 819$, and 606.

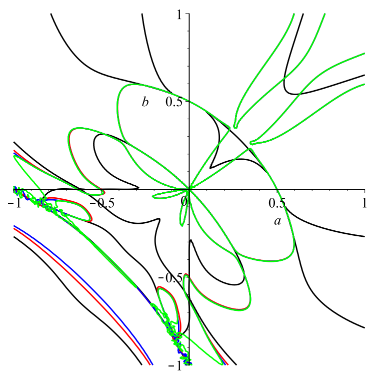
Quartic reversible family (proof II)

Then, the second statement follows just checking that there exists a point (a_0, b_0) in the parameters space such that $P(a_0, b_0) = Q(a_0, b_0) = 0$, $R(a_0, b_0) \neq 0$, $\det \mathrm{Jac}_{(P,Q)}(a_0, b_0) \neq 0$, and $D(a_0, b_0) \neq 0$.
After some tedious work zooming some zones of the figure together with some tricks, we have found a numerical approximation of this special point. Increasing the number of digits in the computations up to see the stabilization of the results, we obtain:

$$
\begin{aligned}
a_0 &\approx 0.62577035826746384070691323127, \\
b_0 &\approx 0.71179266608573393310773491596, \\
R(a_0, b_0) &\approx -1.4439145552036172212169898076\underline{0} \cdot 10^{13}, \\
\det \mathrm{Jac}_{(P,Q)}(a_0, b_0) &\approx -7.7141199535948104150143358564\underline{5} \cdot 10^{29}, \\
D(a_0, b_0) &\approx -9.8789644864239357849860923614\underline{1} \cdot 10^{13}.
\end{aligned}
$$

Quartic reversible family (proof III)

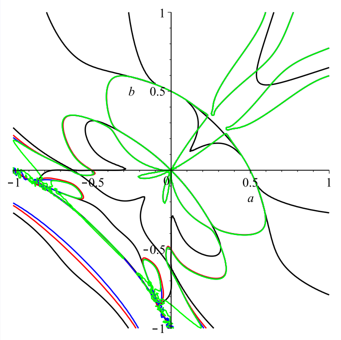
The zero level curves of the polynomials P, Q, R, and D in the square $[-1,1]^2$

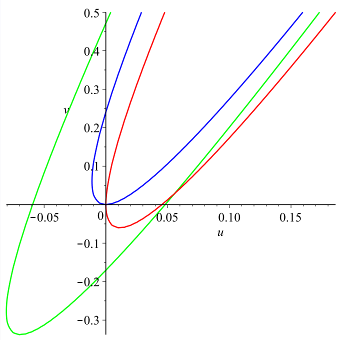


The point (a_0, b_0) should be in the intersection of the red and blue curves but not in the green and black ones, although the curves are very close to see the point.

Quartic reversible family (proof IV)

After a convenient affine change of coordinates we can see that near $(a_0, b_0))$ the zoom is





The complete analytic proof needs to use Poincaré-Miranda Theorem and a computer-assisted proof to check that we have a complete control of the numerics in the intersection of the curves.