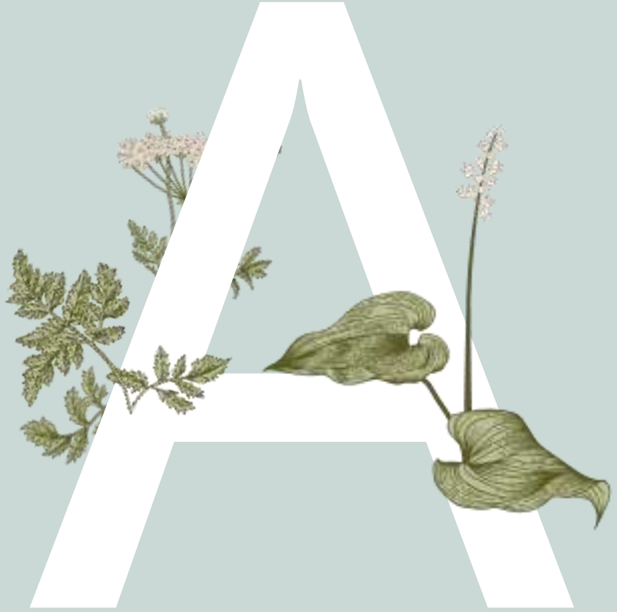
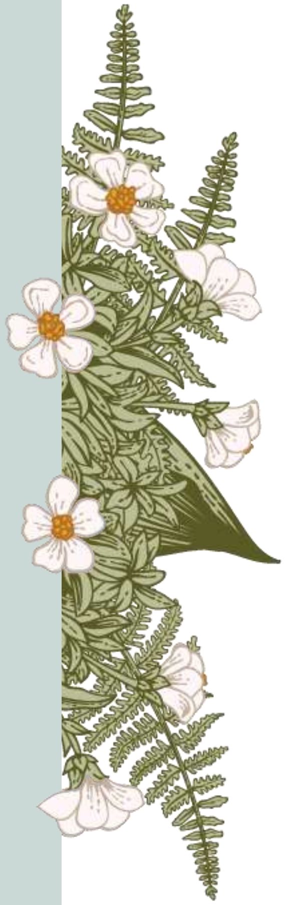
First presentation at the UdL

A new spectral problem and the related integrable nonlinear evolution equations

Reporter: Yang Xiao
Direction: Soliton and Integrable Systems





Objects introduced

Nonlinear evolution equations

Integrable systems

Finite genus solutions

Abstract

A new spectral problem is introduced, which is indicated to be the negative counterpart of the mKdV spectral problem. Based on the fact, some integrable nonlinear evolution equations are obtained, including the derivative Schwarzian KdV equation, the mKdV5 equation and the sine-Gordon equation. Besides, Lax pairs and finite genus solutions of the equations are given.

Nonlinear evolution equations

The derivative Schwarzian KdV equation

$$s_{\tau_{-2}} = \frac{1}{4}\left(s_{yy} - \frac{3s_y^2}{2s}\right)_y ,$$

The mKdV5 equation

$$(\partial_x^2 - \partial_x(lnu) - 4u^2)\left(u_{\tilde{\tau}_1} - u_x\right) = 0,$$

The sine-Gordon equation

$$r_{xy} = 4sinr.$$

Integrable systems

The Hamiltonian functions H_m, H_{-m} are give by

$$H_\lambda = \begin{cases} \lambda - \sum_{m=1}^{\infty} H_m \lambda^{-2m+1}, & |\lambda| > max\{|\lambda_1|, \cdots, |\lambda_N|\} \\ \sum_{m=1}^{\infty} H_{-m} \lambda^{2m-1}, & |\lambda| < min\{|\lambda_1|, \cdots, |\lambda_N|\} \end{cases} \tag{2.8}$$

Where $\lambda_1, \cdots, \lambda_N$ are distinct constants and

$$H_\lambda = [\lambda^2 Q_\lambda(p,q)]^2 + \lambda^2 [1 - Q_\lambda(Ap,p)]\,[1 + Q_\lambda(Aq,q)],$$

$$Q_\lambda(\xi,\eta) = \sum_{j=1}^{N} \frac{\xi_j \eta_j}{\lambda^2 - \lambda_j^2}, \quad A = diag(\lambda_1, \cdots, \lambda_N).$$

Equation (2.8) implies two facts:

(i) H_m and H_{-m} share the same integrals.

H_m is integrable, so H_{-m} is integrable;

further, the linear combination $\widetilde{H}_m = H_m + H_{-1}$ is integrable.

(ii) H_{-m} is the negative counterpart of H_m .

B

From integrable systems to nonlinear evolution equations

The canonical equation of H is

$$\partial_\tau \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\partial H/\partial p \\ \partial H/\partial q \end{pmatrix} \qquad (2.9)$$

where τ denotes the flow variable of H.

Let $H = H_{\pm m}$,$\tau = \tau_{\pm m}$ accordingly, under the constraints

$$u = f^+(p,q) = \langle p, q\rangle; \quad s = f^-(p,q) = \frac{1-\langle A^{-1}q, q\rangle}{H_{-1}}, \qquad (2.10)$$

the j-th component of equation (2.9) is close related to the Lax pair of nonlinear evolution equations.

Example 1

$m = 1$,

$$H_1 = \frac{1}{2}(-\langle p, q\rangle^2 - \langle Aq, q\rangle + \langle Ap, p\rangle),$$

$$\partial_{\tau_1}\begin{pmatrix} p_j \\ q_j \end{pmatrix} = \begin{pmatrix} -\partial H_1/\partial p_j \\ \partial H_1/\partial q_j \end{pmatrix} = \begin{pmatrix} \langle p, q\rangle & \lambda_j \\ \lambda_j & -\langle p, q\rangle \end{pmatrix}\begin{pmatrix} p_j \\ q_j \end{pmatrix}.$$

Denote $\tau_1 = x$, there is the mKdV spectral problem

$$\chi_x = V_1\chi, \qquad V_1 = \begin{pmatrix} u & \lambda \\ \lambda & -u \end{pmatrix}, \ \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}.$$

Example 2

$m=-1$,

$$H_{-1}=\sqrt{[1+\langle A^{-1}p,p\rangle][1-\langle A^{-1}q,q\rangle]},$$

$$\partial_{\tau_{-1}}\begin{pmatrix}p_j\\q_j\end{pmatrix}=\begin{pmatrix}-\partial H_{-1}/\partial p_j\\ \partial H_{-1}/\partial q_j\end{pmatrix}=\begin{pmatrix}0 & \lambda_j^{-1}[1+\langle A^{-1}p,p\rangle]\\ \lambda_j^{-1}[1-\langle A^{-1}q,q\rangle] & 0\end{pmatrix}\begin{pmatrix}p_j\\q_j\end{pmatrix}.$$

Denote $\tau_{-1}=y$, there is the spectral problem(the negative counterpart of 1)

$$\chi_y=V_{-1}\chi,\quad V_{-1}=\begin{pmatrix}0 & \lambda^{-1}s^{-1}\\ \lambda^{-1}s & 0\end{pmatrix},\chi=\begin{pmatrix}\chi_1\\\chi_2\end{pmatrix}.$$

Lax pair and equations

Lax Pair: $\chi_x = V_1\chi, \qquad \chi_{\tau_m} = V_m\chi$

The Compatibility condition

$$\chi_{x\tau_m} = \chi_{\tau_m x}$$

gives

$$(V_1)_{\tau_m} = (V_m)_x - [V_1, V_m].$$

From which some interesting equations are obtained.

Example 3: $m = 2$

Lax Pair: $\chi_x = V_1\chi, \quad \chi_{\tau_2} = V_2\chi\,,$

$$V_2 = \begin{pmatrix} \lambda^2 u + \frac{1}{2}u^3 + uu_x + \frac{1}{4}u_{xx} & \lambda^3 + \frac{\lambda}{2}(u^2 - u_x) \\ \lambda^3 + \frac{\lambda}{2}(u^2 + u_x) & -\left(\lambda^2 u + \frac{1}{2}u^3 + uu_x + \frac{1}{4}u_{xx}\right) \end{pmatrix}$$

Equation: $u_{\tau_2} = -\frac{3}{2}u^2 u_x + \frac{1}{4}u_{xxx}$

Example 4

Lax Pair: $\chi_y = V_{-1}\chi$, $\chi_{\tau_{-2}} = V_{-2}\chi$;

Equation: the derivative Schwarzian KdV equation

$$s_{\tau_{-2}} = \frac{1}{4}\left(s_{yy} - \frac{3s_y^2}{2s}\right)_y.$$

Example 5

Lax Pair: $\chi_x = V_1\chi$, $\chi_{\tilde{\tau}_1} = \tilde{V}_1\chi$, $\tilde{V}_1 = V_{-1} + V_1$;

Equation: the mKdV5 equation

$$(\partial_x^2 - \partial_x(lnu) - 4u^2)\left(u_{\tilde{\tau}_1} - u_x\right) = 0.$$

Example 6

Lax Pair: $\chi_y = V_{-1}\chi,\qquad \chi_x = V_1\chi;$

Equation: $u_y = s^{-1} - s,\ \ -2u = \partial_x(lns).$

By eliminating u in the above equation, we have

$$(lns)_{xy} = -2(s^{-1} - s).$$

Let $r = -\sqrt{-1}lns$, then r satisfies the sine-Gordon equation:

$$r_{xy} = 4sinr.$$

Finite genus Solutions

If $\zeta = \lambda^2$ and $\alpha(\zeta) = \prod_{j=1}^{N}(\zeta - \lambda_j^2)$, then there are some decompositions

$$H_\lambda^2 = \frac{R(\zeta)}{\alpha^2(\zeta)}, \quad R(\zeta) = \zeta \prod_{j=1}^{2N}(\zeta - \lambda_j^2); \quad 1 - Q_\lambda(Ap, p) = \frac{n(\zeta)}{\alpha(\zeta)}, \quad n(\zeta) = \prod_{j=1}^{g}(\zeta - v_j^2); \quad 1 + Q_\lambda(Aq, q) = \frac{m(\zeta)}{\alpha(\zeta)}, \; m(\zeta) = \prod_{j=1}^{g}(\zeta - \varepsilon_j^2). \quad (4.1)$$

The algebraic curve is defined by $\Gamma\colon \xi^2 - R(\zeta) = 0$, with the genus $g = N$. On Γ, the normalized holomorphic differential is

$$\omega = \frac{1}{4\sqrt{R(\zeta)}}\left(C_1 \zeta^{g-1} + \cdots + C_g\right), \quad (4.2)$$

$C_1, \cdots C_g$ are constants. Let P_0 be a fix point on Γ, then the Abel-Jacobi coordinates are defined by

$$\phi = \sum_{j=1}^{g} \int_{P_0}^{P(v_j^2)} \omega, \quad \psi = \sum_{j=1}^{g} \int_{P_0}^{P(\varepsilon_j^2)} \omega, \quad (4.4)$$

$\{v_j^2\}$ and $\{\varepsilon_j^2\}$ provide the elliptic coordinates.

Example 7

$g = 1$

The algebraic curve Γ:

$$\xi^2 - R(\zeta) = 0,\ R(\zeta) = \zeta(\zeta - \lambda_1^2)\,(\zeta - \lambda_2^2).$$

Then equations (4.2) and (4.4) present two elliptic equations

$$(dv_1^2/d\phi)^2 = 16C_1^{-2}\ R(\zeta),\ \ (d\varepsilon_1^2/d\psi)^2 = 16C_1^2\ R(\zeta),$$

which can be solved directly by elliptic functions in several forms of different parameters. Now we chose one as an example.

Example 7$(g = 1)$

Let $A = \lambda_1^2$, $m^2 = \lambda_1^2/\lambda_2^2$, $k^2 = 4\lambda_2^2/C_1^2$, then we have solutions

$$v_1^2 = ASn^2(k(\phi - \phi_0), m),\ \varepsilon_1^2 = ASn^2(k(\psi - \psi_0), m).$$

Specially, let $m^2 = 1$, $b = -8k^2 = -32\lambda_2^2/C_1^2$, $c = \frac{4k^2}{A} = 16/C_1^2$, the above solutions turns into

$$v_1^2 = -\frac{b}{2c} tanh^2\left(\frac{-b}{2\sqrt{2}}(\phi - \phi_0)\right),\ \varepsilon_1^2 = -\frac{b}{2c} tanh^2\left(\frac{-b}{2\sqrt{2}}(\psi - \psi_0\phi_0)\right).$$

Based on (2.10) and (4.1), the relation holds: $u^2 = 2\lambda_1^2 - 2H_1 - (v_1^2 + \varepsilon_1^2)$**.** Thus the explicit solution of e mKdV5 equation is given by

$$u = \pm\sqrt{2\lambda_1^2 - 2H_1 + \frac{b}{2c} tanh^2\left(\frac{-b}{2\sqrt{2}}(x\Omega_1 - C_2\Omega_2)\right)}$$

Thanks

Yang Xiao

Xiao.Yang@udl.cat

yx@zzu.edu.cn